THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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VOL. XIII. MARCH, 1927. No. 187.

The Mathematical Association.

THE Annual Meeting of the Mathematical Association was again held at the London Day Training College.

On 3rd January Professor R. W. Genese entertained an attentive audience with "A Simple Exposition of Grassmann's Methods."

On 4th January, after the Report of the Council had been received, a few Rules had been altered, and other business had been transacted, Professor M. J. M. Hill delivered his Presidential Address, "On the Teaching of Mathematics." The President was under doctor's orders, and after carrying through this task with evident difficulty but with complete success, he asked to be excused from attending the further proceedings, and left the Meeting, acknowledging at the doorway the acclamation by which members showed their appreciation of his perseverance. Professor Alfred Lodge was asked to take the Chair, and although he vacated it at once in order to read a paper on his own "Graphic Solution of Quadratic Equations," he afterwards occupied it until the end of the Meeting.

Two lively discussions took place in the course of the day. The first, on "The Choice of Units in the Teaching of Mechanics," was opened by Mr. G. Goodwill; the second followed a paper by Professor H. F. Baker with the challenging title "Can the Range of Geometry taught in Schools be widened?" Papers by Mr. W. C. Fletcher on "Geometrical Congruence," by Mr. A. C. Heath on "The Approximation to Irrational Numbers by Rationals," and by Mr. A. Trystan Edwards on "The Relation of Art to Mathematics," were the other items in an excellent programme.

REPORT OF THE COUNCIL FOR THE YEAR 1926.

DURING the year 1926, 98 new members of the Association have been admitted, and the number of members now on the Roll is 1130. Of these, 8 are honorary members, 53 are life members by composition, 22 are life members under the old rule, and 1047 are ordinary members. The number of Associates is about 550.

The Council regret to have to record the deaths of Mr. T. S. Barrett, of Widnes; Mr. R. F. Duckworth, Headmaster of St. Peter's School, Weston-super-Mare; Mr. Vivian Le Neve Foster, formerly of Eton College; Mr. E. P. Rouse, formerly of Eton College; and the Rev. E. W. P. Temperley, of St. Paul's School.

Early in the year the *Report on the Teaching of Mathematics to Evening Technical Students* and a *List of Books suitable for School Libraries* were issued to members. References to these were made in the Council's report for 1925. The Honorary Librarian has also prepared a *Catalogue of Books and Pamphlets acquired before the year* 1924, and now in the library of the Association. A supplement is in course of preparation. The Catalogue, together with a List of those members of the Association who were on the Roll on 30th June, 1926, was issued to members during the year.

A *Report on Mathematics in Examinations for Entrance Scholarships in Public Schools* has been prepared by the Boys' Schools Committee, and has been issued to members.

A special volume will be issued early in 1927, commemorative of the bicentenary of the death of Sir Isaac Newton, which occurred on 20th March, 1727.

Mr. C. O. Tuckey has been appointed Chairman, and Mr. A. Robson, Secretary, of the new General Teaching Committee; Mr. C. O. Tuckey, Chairman, and Mr. F. C. Boon, Secretary, of the Boys' Schools Committee; Miss M. J. Griffith, Chairman, and Miss M. A. Hooke, Secretary, of the Girls' Schools Committee. Mr. W. J. Dobbs is again Secretary of the Examinations Sub-Committee.

The Council have the pleasure of nominating Professor Gino Loria, D.Sc., Laureat de l'Institut de France, for election as an honorary member of the Association. Dr. Loria is Professor of Higher Geometry in the University of Genoa, and has been an ordinary member of the Association since 1894.

As it will be necessary to reprint the Rules of the Association immediately, the Council will lay before the Annual Meeting proposals for a few changes which the experience of recent years and the development of the Association suggest as desirable and necessary.

The Council desire to express their warmest thanks to Mr. A. Dakin and Mr. C. E. Williams, who retire from the Council in accordance with the rules, for the valuable services they have rendered to the Association during their tenure of office as members of the Council

since 1922 and 1908 respectively. From 1908 to 1921 Mr. Williams also acted as Honorary Librarian.

The thanks of the Association, and especially of those members who make use of the Library, are owing to Mr. F. Beames, formerly of Bristol Grammar School, who has voluntarily undertaken, since last June, the least attractive part of the routine work of the Library.

The Council desire also to acknowledge the indebtedness of the members of the Association to Mr. W. J. Greenstreet for the continuance of his excellent work as Editor of the *Mathematical Gazette*.

ALTERATION OF RULES.

The following changes were proposed by the Council and agreed to:

Rule V.—The first paragraph, 2nd line: The last three words to be altered thus: "And the number of such elections."

In the second paragraph, after the first sentence, the following words to be inserted: "in return, each member shall receive one copy of each issue of *The Mathematical Gazette*, and of any other publication issued during the year."

To be added to the last paragraph:

"After continuous membership for 20 years, the payment for Life Composition shall be reduced to Five Guineas."

Rule VI.—The third paragraph to read thus:

"An Associate shall be entitled to attend all meetings of the Association and to use the Library, but shall not be entitled to receive any of the publications of the Association or to vote on any matter of business."

The fourth paragraph to read thus:

"Each branch shall pay to the Association an annual subscription of fifteen shillings; and, in return, shall receive one copy of each issue of *The Mathematical Gazette* and of any other publication issued during the year, but any Branch may pay more than one such subscription of fifteen shillings, and shall receive in return a corresponding number of copies of each Gazette and of each other publication."

In the sixth paragraph, 3rd line, after the word "Branch," the words "who are not in arrears with their subscriptions to the Branch," to be inserted.

In the seventh paragraph, 9th line, after the word "Association," the words "and shall not be paid on account of any member who is in arrear with his subscription to the Branch," to be inserted.

Rule VII.—The word "Honorary" to be inserted in five places.

Rule X.—The second paragraph to be omitted.

Rule XI.—From the second paragraph the words "which shall be presented at the Annual General Meeting," to be omitted.

Rule XIV.—The third paragraph to read thus:

"The Treasurer shall report on the financial position of the Association."

SIR GEORGE GREENHILL

November 29, 1847—February 10, 1927

President of the Mathematical Association, 1913-1914

ON THE TEACHING OF MATHEMATICS.

Presidential Address to the Mathematical Association, 1927.

By Professor M. J. M. Hill.

Introduction.

Ladies and Gentlemen,—I desire to begin by thanking you for the great honour you have conferred on me by inviting me to occupy the Presidential Chair of your Association.

Great as is the honour, I am thinking more of the responsibilities it entails and of the opportunities for useful work which it offers to me.

The noble passage which stands at the head of the statement of the objects of this Association opens with the words: "I hold every man a debtor to his profession." The consciousness of that debt I carry with me whether at work or in repose.

Thus it happened that when some years ago I was spending a holiday on the shore of the North Sea, I was thinking of my profession. And whilst I was so engaged there came and sat down beside me a little lad. I asked him if he went to school. "Yes!" he said indifferently. Though somewhat repelled, I ventured on another question: Did he like his school? "No!" he answered, pouring into the word a depth of indignation which stung me to the quick, for I felt its implied censure on my profession. And as I moved uneasily away over the stones, my eyes fell upon a fragment of paper which contained a paragraph headed: "She knew her multiplication table." It read something like this: Little Sue had come home from school. She ran up to her mother and said, "Mother, I know my multiplication table." Her mother said, "Hush, Sue, Uncle Bob is reading. Don't disturb him." But her excitement was not to be quenched, so she answered, "Did you hear, Mother, what I said? I know my multiplication table." This roused Uncle Bob, who said, "Did you say you knew your multiplication table, Sue?" "Yes! Uncle," she replied. "Very well, then," he said, "tell me what are thirteen times three?" Whereupon she looked at him scornfully and said, "Don't be so silly, Uncle Bob, there ain't no such thing as thirteen times three."

Some time after this, when I was thinking what significance this had for those interested in the efficiency of education, the scene before my eyes seemed to change. It was a summer afternoon. There was great excitement in the schoolroom. The air was quivering with heat; a distinguished ecclesiastic had just entered the room to address the lads. I was listening intently, and I seemed to hear these words: "We owe a great debt to the Headmaster for the way in which he has met the Philistine attack of those who are attempting to put down Classics in favour of such plebeian pursuits as the study of Mathematics."

Plebeian! How strange an epithet to apply to the study of any branch of human thought. Can you wonder that in my indignation there flashed through my mind a distortion of a well-known saying so that it read thus: "Omne ignotum pro *malefico*." But a gibe carries one no farther, and reflection brought a wiser counsel, so that I remembered: "Fas est et ab hoste doceri," and I was forced to put some questions to myself.

Have we teachers of mathematics, you and I, failed to strike a responsive chord in the minds of our pupils, whether in the public or in the elementary schools? Is there anything beyond the inherent difficulty of the subject that stands between them and us?

I think there is. Driven by the great mass of routine, we teach, I fear, too much by rule, too little by principle. Let me try to make my meaning quite clear. I use the word "principle" as something furnishing the pupil

with a direction which commends itself immediately to his reason, something which he can link up at once with his previous experience. He welcomes it gladly. On the other hand a "rule" is of the nature of an order; very often it is a summary of reasons which may either have been clearly and fully explained, or which may have been inadequately explained, or which may have been delivered *ex cathedrâ* without any explanation at all.

Possibly an objection may be made at this point. The teacher is obliged to teach the subject so that the pupil can acquire great facility in using it as a tool, and this can only be done by the formulation of rules. Unfortunately, facility is often acquired at the price of forgetfulness of the reasons on which these rules were founded. This price is too great to pay. My suggestion is that in every lesson a few moments should be spared to go back to first principles, for whenever these have been fully grasped the desired facility will follow by itself.

And so the plan of my address from this point onwards is to choose out two or three rules or statements commonly made without explanation, or as it seems to me with inadequate explanation, and to suggest some line of action which has helped me in the past in the hope that it may help some of you in the future.

The Names of Numbers.

It must be admitted at the outset that we teachers of mathematics suffer from a serious handicap, for which, not we, but our forbears are responsible; I refer to the names of the numbers from eleven to nineteen. Have you ever watched a child hesitate in naming numbers in this group? Just as "twenty-one" suggests two tens and one, so "eleven" should suggest one ten and one: just as "twenty-three" suggests two tens and three, so "thirteen" should suggest one ten and three; but the two syllables of the word "thirteen" are in the wrong order. The first syllable should suggest ten and the second three; and this same inversion of order continues until "nineteen" is reached. It is no consolation to observe that the French are in still worse plight, for not only have they the same kind of difficulty with the numbers eleven to sixteen, but, except in the south of France, in Switzerland, and in some parts of Belgium, they have abandoned the very convenient terms, derived from Latin, "septante" for seventy, "octante" for eighty, and "nonante" for ninety, adopting in their place soixante-dix representing $60+10$ for seventy, quatre-vingt representing 4×20 for eighty, and quatre-vingt-dix representing $(4\times20)+10$ for ninety.

Obviously here the reformers who established the decimal system quailed before the task of reducing these names to reasonable dimensions.

Brackets.

It may be that some of you wonder why I have placed a bracket in the expression $(4\times20)+10$.

I was once taken to task, not many years ago, for not knowing that the expression

$$a+b\times c$$

had been ruled to mean

$$a+(b\times c)$$

and not

$$(a+b)\times c.$$

Against this rule I enter a most solemn protest. The strain upon the memory in mathematical work is so great that it is inexcusable to make any rule which is not necessary. The desire to save oneself the trouble of writing a bracket is not a sufficient excuse. And in the present case the alleged rule, made by some unknown authority, is contrary to the system of reading universally adopted in western countries, where one reads from left to right.

I hold that the child, who, being asked what meaning he would attach to the expression, $a+b\times c$, and who answers $(a+b)\times c$, is a much more intelligent being than one who says that it means $a+(b\times c)$, because the first brings what is put before him into relation with his previous experience of reading from left to right, whilst the second does not.

Perhaps you may reply that $a+bc$ means $a+(b\times c)$.

I agree, but in $a+bc$ the bracket is implied and "bc" is a single expression, whereas in $a+b\times c$ the letters b and c are separately written.

Division.

My next comment is upon the rules employed in the case of the division of one number by another, and especially in the case of the division of one decimal by another.

This is how I was taught in the dim and distant past of the early Victorian era, and this is how I have reason to believe many children are taught now.

You will no doubt find what I propose in some book. I have read a good many treatises on arithmetic, and also the official publications by the Board of Education of the English * papers read at the last International Mathematical Education Congress, but I have not found anywhere exactly what I suggest *in the case of the division of decimals.* But if my suggestion is an old one, as I believe it must necessarily be, I shall be grateful to any member of this Association who will send me the reference.

To proceed, let us suppose that we desire to divide 8397 by 29, by which we mean now that it is desired to find out how many times 29 can be subtracted from 8397. De Morgan begins his explanation of division by taking quite small numbers and taking away the divisor once from the dividend, then once again, and so on.

This, however, is the way in which I was taught:

```
29 ) 8397 ( 289 Quotient.
     58
     ---
     259
     232
     ---
      277
      261
      ---
       16 Remainder.
```

That is the scheme. Some explanation is given, over which I need not stop.

Instead, I put this question to you.

When the child divides 83 by 29, does he consciously realise that the 2 in the quotient 289 means 200, and that he is really subtracting 200 times the divisor from the dividend?

At the next step he is told to bring down the 9, just as if it were a bale of goods which he was told to carry into the next room. This is not an arithmetical operation. Does he realise at the next step that he is subtracting 80 times the divisor from 2590, and so on?

I suggest to you that at the beginning it would be better to follow this other scheme:

```
29 ) 8397 |
     5800 | = divisor × 200
     ----
     2597
     2320 | = divisor ×  80
     ----
      277
      261 | = divisor ×   9
      ---   ---------------
       16 |             289 Quotient.
```

* But not the foreign ones.

Does it not stare one in the face that what one would then be doing is to subtract successive multiples of the divisor first from the dividend and then from the remainders left after each division ?

There is still another question. How is it to be decided that the first figure in the quotient means 200 ? The answer to this contains the germ of all that follows here regarding division.

I begin by forming mentally or otherwise the multiples of the divisor by 1, 10, 100, 1000 ... until I find two of these between which the dividend lies:

$$29 \times 100 = 2900, \quad 29 \times 1000 = 29000.$$

As the dividend falls between these, it follows that it is possible to subtract 100 twenty-nines, but not 1000 twenty-nines from the dividend.

Therefore the quotient lies between 100 and 1000.

I form next the multiples of the divisor by

$$100, 200, 300, \ldots, 900,$$

and then I find that the dividend lies between

$$29 \times 200 \quad \text{and} \quad 29 \times 300.$$

So it is possible to subtract twenty-nine 200 times, but not 300 times, from the dividend.

Therefore the quotient lies between 200 and 300.

Hence the first figure in the quotient represents 200.

Not until this had been very completely mastered would I venture to give the abbreviated form from which I set out at first, and I should take many opportunities of finding out whether the process were really remembered and understood.

Multiplication of Decimals.

Before passing on to the division of one decimal by another, I premise that it is essential that the beginner should be taught to *reason out* the value of the product of a single digit in any place by a single digit in any place.

Thus
$$\cdot 02 \times \cdot 003 = \frac{2}{10 \times 10} \times \frac{3}{10 \times 10 \times 10} = \frac{6}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= \cdot 00006.$$

$$\cdot 04 \times \cdot 007 = \frac{4}{10 \times 10} \times \frac{7}{10 \times 10 \times 10} = \frac{28}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= \frac{20}{10 \times 10 \times 10 \times 10 \times 10} + \frac{8}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= \frac{2}{10 \times 10 \times 10 \times 10} + \frac{8}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= \cdot 0002 + \cdot 00008$$
$$= \cdot 00028.$$

Division of Decimals.

I am now in a position to take up the question of dividing one decimal by another.

May I ask you to look at De Morgan's *Arithmetic*. He offers five methods of determining the position of the decimal point in the quotient. Plainly we have here an *embarras de richesses*.

One of these he calls the common rule, but says that it is defective in that it does not apply in all cases. Therefore we will pass it over.

In two others he directs that very useful officer the decimal point to stand aside, whilst he performs certain manœuvres with his subordinates, and then he orders him back to a certain place. This procedure often leads to such

confusion in the mind of the beginner that the decimal point fails to reach the right position. In each of these two cases the original division is replaced by another which has the same quotient but a different remainder. Therefore we will pass these by also.

In a fourth he tells the beginner to make himself familiar with the rule of the characteristic, of which I preserve a dolorous recollection, and which he explains in rather more than a page of his Appendix. No doubt this rule must be mastered by the practical engineer, but he should have the basis of it clearly explained to him. This can be done thus:

It should be first explained that any number can be expressed either as the product of a number having the same digits in the same order as itself whose value lies between 1 *inclusive* and 10 *exclusive* (conveniently called a standard number) and a power of 10; or else as the quotient of such a number by a power of 10; thus

$$23{\cdot}796 = 2{\cdot}3796 \times 10;$$
$$\cdot 023796 = 2{\cdot}3796 \div (10 \times 10).$$

This is very convenient in the expression of very large and very small numbers, and is in common use in treatises on Physics and Chemistry.

Let us now represent digits by letters, then $a{\cdot}bc\ldots$ and $e{\cdot}fgh\ldots$, where a and e do not vanish, are standard numbers.

There are three cases to consider:

(i) If $a{\cdot}bc\ldots > e{\cdot}fgh\ldots,$

then $$1 < \frac{a{\cdot}bc\ldots}{e{\cdot}fgh\ldots} < 10;$$

$$\therefore \frac{a{\cdot}bc\ldots}{e{\cdot}fgh\ldots} = k{\cdot}lmn\ldots.$$

where k does not vanish.

(ii) If $a{\cdot}bc\ldots = e{\cdot}fgh\ldots,$

then $$\frac{a{\cdot}bc\ldots}{e{\cdot}fgh\ldots} = 1.$$

(iii) If $a{\cdot}bc\ldots < e{\cdot}fgh\ldots,$

then $$\cdot 1 = \frac{1}{10} < \frac{a{\cdot}bc\ldots}{e{\cdot}fgh\ldots} < 1.$$

In this case $$\frac{a{\cdot}bc\ldots}{e{\cdot}fgh\ldots} = {\cdot}rst\ldots,$$

where r does not vanish.

To these three cases all others can be reduced. What more remains is only the multiplication or division by some power of 10.

For example:
$$\frac{23{\cdot}876}{{\cdot}129} = \frac{2{\cdot}3876 \times 10}{1{\cdot}29 \div 10}$$
$$= \frac{2{\cdot}3876}{1{\cdot}29} \times 10 \times 10$$
$$= 1{\cdot}abc\ldots \times 10 \times 10 = 1ab{\cdot}c\ldots.$$

But
$$\frac{{\cdot}129}{23{\cdot}876} = \frac{1{\cdot}29 \div 10}{2{\cdot}3876 \times 10}$$
$$= \frac{1{\cdot}29}{2{\cdot}3876} \times \frac{1}{10 \times 10} = \frac{{\cdot}5ab\ldots}{10 \times 10}$$
$$= {\cdot}005ab\ldots.$$

It will be seen that there is no necessity to remember any rule. All that is necessary is to remember how to express any number as the product of the standard number which has the same digits in the same order as itself and a power of 10, or else as the quotient of this standard number by a power of 10.

This is one of the two methods for determining the position of the decimal point in a quotient which I believe not to be open to objection. It will be found of the greatest use when the beginner has to deal with logarithms.

But there is still a fifth method offered by De Morgan. He says: "The student is recommended ... to *reason out* the place of the decimal point. Thus, it should be visible, that

$$26{\cdot}119 \div 7{\cdot}2436$$

has one figure before the decimal point, and that $26{\cdot}119 \div 724{\cdot}36$ has one cipher after it, preceding all significant figures."

And when I had got so far, I laid down the book disheartened. I felt—Well, you remember what you felt when you followed Childe Roland to the Dark Tower, and you saw him place the slug-horn to his lips. Then it seemed that the secrets of the Dark Tower were about to be revealed. But, No! when the last blast of the horn had told you that Childe Roland had come to the Dark Tower, a deathly silence fell upon the scene.

So now, with my expectations raised to the highest pitch, my hopes were dashed to the ground, and I turned back to think, and this, for what it is worth, is the result of my thoughts.

Just as in division by a whole number we subtract successive multiples of the divisor from the dividend, so here we have to subtract multiples of tenths, hundredths, thousandths, ... of the divisor from the dividend. That is the key to the Dark Tower.

And we can discover the position of the decimal point thus:

Multiply * the divisor not only by 10, 100, 1000 ... as already explained, but also by ·1, ·01, ·001, ... until we find two products which contain the dividend between them.

Suppose it is found that the dividend lies between the products of the divisor by ·01 and ·001.

Then we know that the first figure in the quotient is one of the following:

$$\cdot001,\ \cdot002,\ \cdot003,\ \cdot004,\ \cdot005,\ \cdot006,\ \cdot007,\ \cdot008,\ \cdot009.$$

Let me take De Morgan's second example.

To divide 26·119 by 724·36.

We do not trouble to multiply the divisor by 1, 10, 100, ..., as we should in each case get a product greater than the dividend; but we multiply the divisor by ·1, ·01, ·001, etc.

Thus
$$724{\cdot}36 \times \cdot1 = 72{\cdot}436;$$
$$724{\cdot}36 \times \cdot01 = 7{\cdot}2436.$$

Obviously we need go no further.

The quotient is between ·1 and ·01.

Consequently we form the products of the divisor by ·01, ·02, ·03, ... ·09:

$$724{\cdot}36 \times \cdot01 = 7{\cdot}2436;$$
$$724{\cdot}36 \times \cdot02 = 14{\cdot}4872;$$
$$724{\cdot}36 \times \cdot03 = 21{\cdot}7308;$$
$$724{\cdot}36 \times \cdot04 = 28{\cdot}9744.$$

We need go no further.

The quotient is between ·03 and ·04.

Hence the beginning of the quotient is ·03.

* I use the words 'multiply' and 'times' in an extended sense for the sake of brevity. I should not do this in actual teaching without explanation.

The work can be set down thus:

$$\begin{array}{r|l} 724{\cdot}36\ )\ 26{\cdot}119 & \\ \underline{21{\cdot}7308} & = \text{divisor} \times {\cdot}03 \\ 4{\cdot}3882 & \end{array}$$

We proceed in like manner to divide 4·3882 by the divisor, and we find that it lies between the divisor multiplied by ·006 and the divisor multiplied by ·007.

The work can be set out thus:

$$\begin{array}{r|l} 724{\cdot}36\ )\ 26{\cdot}119 & \\ \underline{21{\cdot}7308} & = \text{divisor} \times {\cdot}03 \\ 4{\cdot}3882 & \\ \underline{4{\cdot}34616} & = \text{divisor} \times {\cdot}006 \\ {\cdot}04204 & \end{array}$$

We proceed in like manner to divide ·04204 by the divisor. At this step the beginner is liable to be perplexed. He sees that the next significant figure is 5, and he tries to see if it represents ·0005.

Now $724{\cdot}36 \times {\cdot}0005 = {\cdot}362180$.

This is too much.

He tries, therefore, $724{\cdot}36 \times {\cdot}00005 = {\cdot}0362180$.

And the work can be set out thus:

$$\begin{array}{r|l} 724{\cdot}36\ )\ 26{\cdot}119 & \\ 21{\cdot}7308 & = \text{divisor} \times {\cdot}03 \\ \underline{4{\cdot}3882} & \\ 4{\cdot}34616 & = \text{divisor} \times {\cdot}006 \\ \underline{{\cdot}04204} & \\ {\cdot}0362180 & = \text{divisor} \times {\cdot}00005 \\ \underline{{\cdot}0058220} & \underline{\qquad\qquad {\cdot}03605\ \text{Quotient.}} \end{array}$$

And now it leaps to the eye that the divisor has been multiplied successively by ·03, ·006, ·00005, and that these products have been subtracted from the dividend and the successive remainders, so that we have really multiplied the divisor by ·03605 and subtracted that product from the dividend; the remainder then left is ·0058220.

You will observe that I have not hurt the feelings of the decimal point by ordering him about as if he were a mere cipher, a man of no account. This is the second method of determining the position of the decimal point in a quotient which I believe is not open to objection.

I may be met with the objection that there is no time for all this. Mr. Inspector and Mr. Examiner are standing close at hand demanding swift results. Well! it seems to me that it is more likely that they will be satisfied if, as I have already suggested, some portion of every lesson is given up to explanations such as the above, and the bulk of the time will still remain in which to give to the pupils that facility in the rest of the routine of handling numbers which it is essential that they should possess.

The Postulate of Parallels.

I pass now to one of the most striking instances of teaching by rule known to me. I refer to Euclid's Postulate of Parallels. It appears in the text of his *Elements* without any preliminary explanation like a bolt from the blue, and the student may well enquire how to link it with his previous experience.

I am not now raising issues with the sole purpose of simplifying the subject for beginners, but with the view of asking your attention to matters which seem to me to be of importance in the orderly development of the subject.

And it is specially important at this moment in view of the following extract from the preface to the second edition of Sir T. L. Heath's edition of the *Elements*:

"I refer to the movement in favour of reviving, in a modified form, the proposal made by Wallis in 1663 to replace Euclid's Parallel-Postulate by a Postulate of Similarity. ... The form of Postulate now suggested is an assumption that 'Given one triangle, there can be constructed ... another triangle equiangular with (or similar to) the given triangle.' It may perhaps be held that this assumption has the advantage of not referring, in the statement of it, to the fact that a straight line is of unlimited length; but, on the other hand, as is well known, Saccheri showed (1733) that it involves more than is necessary to enable Euclid's Postulate to be proved. In any case it would seem certain that a scheme based upon the proposed Postulate, if made scientifically sound, must be more difficult than the procedure now generally followed. This being so, and having regard to the facts (1) that the difference between the suggested Postulate and that of Euclid is in effect so slight and (2) that the historical interest of Euclid's Postulate is in fact so great, I am of opinion that the proposal is very much to be deprecated."

On this I remark:

(1) I am greatly puzzled by the reference to Saccheri. A German translation of Saccheri's great work, *Euclides ab omni naevo vindicatus*, will be found in Engel and Stäckel's *Theorie der Parallel-linien*.* The passage referred to by Sir T. L. Heath is on pp. 84-5. But there is a footnote attached pointing out that the proof is essentially defective, since Saccheri used the property of the triangle that the exterior angle is greater than either of the interior and opposite angles (Euc. I. 16). The employment of this proposition is inconsistent with Saccheri's Hypothesis of the Obtuse Angle; and, further, the assumption of the existence of two similar triangles is by itself sufficient to exclude the Hypothesis of the Acute Angle as well as that of the Obtuse Angle.

It would seem therefore that this objection falls to the ground.†

(2) The next comment I have to make is on the words: "It seems certain that a scheme based on the proposed Postulate, if made scientifically sound, must be more difficult than the procedure now generally followed." This is the crux of the whole matter.

I remind you that Euclid's theory rests *inter alia* on the postulate that all right angles are equal, and on the proposition, or (as it is now generally regarded) the postulate that if two of the sides and the included angle of a triangle are respectively equal to the corresponding elements of another triangle, then the triangles are congruent (Euc. I. 4).

Consequently Euclid's demonstration of the theory of parallels depends on the idea of the congruence of *areas*, an idea which is foreign to the subject of that theory, which is concerned only with straight lines and their intersections, and with angles, *not with areas*.

It is of great importance to the teaching of Mathematics that the issues involved should be examined with care.

As my contribution to the discussion I place before the Association a theory of parallels based on a modified form of Wallis's Postulate, which assumes *neither that all right angles are equal nor the idea of the congruence of areas;* and I deduce from the theory that all right angles are equal. It is to be

*Saccheri's work is divided into two books, the first dealing with parallels, the second with proportion. An English translation of the first book, by G. B. Halsted, with a reprint of the Latin original, was published in 1920 by the Open Court Company.—[Ed.]

†Although Saccheri's proof that the existence of two non-congruent similar triangles involves the truth of the Hypothesis of the Right Angle is not valid, yet valid proofs exist. But it is to be noted that the deduction of Euclid's Postulate of Parallels from Saccheri's Hypothesis of the Right Angle requires assumptions not needed in the Theory given below. These assumptions are: (1) The Postulate that all right angles are equal; (2) The Propositions or Postulates involved in the theory of the congruence of areas.

noticed that I have not given Euclid's proposition for constructing an angle equal to a given angle. The actual construction, which requires the theory of the congruence of areas, is not necessary to the demonstration of the theory.

The theory given below is complete in itself and forms a section of Geometry independent of all other geometrical propositions. From the logical point of view it would be possible to commence the teaching of Geometry with it.

Following Hilbert I make no attempt to define points, straight lines, or planes. Hilbert's interpretation of the word " between " will be found in his *Foundations of Geometry*, pp. 5-7.

THEORY OF PARALLEL STRAIGHT LINES AND RIGHT ANGLES

BASED ON A MODIFIED FORM OF WALLIS'S POSTULATE OF SIMILARITY.

DEFINITIONS.

1. A ray is the part of a straight line bounded by one of its points. The point which bounds the ray is called its origin.

2. An angle is a plane figure bounded by two rays having the same origin.

3. If the rays which bound an angle lie on the same straight line, but in opposite directions, the angle is called a straight angle or a flat angle.

4. Two straight lines are parallel if they lie in the same plane but do not meet.

AXIOMS.

(i) Any two points determine a straight line.

(ii) Any three points in space, which do not lie on a straight line, determine a plane.

(iii) If there be a ray OX in a plane, then two rays OY, OZ exist in this plane, each of which makes with OX an angle equal to a given angle.

(iv) If two points A, B of a straight line a lie in a plane, then every point of a lies in the plane.

(v) If ABC be a triangle, and a ray be drawn from A to a point inside the triangle, this ray meets the side BC in some point between B and C.

(vi) If $A=B$ and $B=C$, then $A=C$.

(vii) If $A=B$ and $C=D$, then $A+C=B+D$.

(viii) If $A+B<C+D$ and if $A=C$, then $B<D$.

POSTULATE.

This is a modified form of Wallis's Postulate,* and may be stated thus:

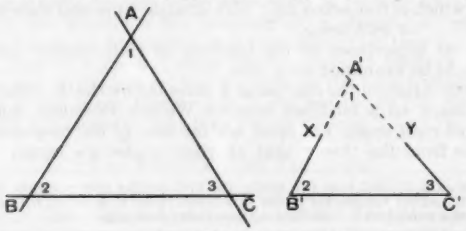


FIG. 1.

* Wallis stated his Postulate thus: "To every figure there exists a similar figure of arbitrary magnitude." Although in the form to be used there is no reference to similarity or to the idea of ratio which similarity suggests, it will be convenient to refer to the modified Postulate as Wallis's.

Let there be three straight lines in a plane which intersect so as to form a triangle ABC.

Take any two points B', C'.

Then a ray $B'X$ exists which passes through B' and makes with $B'C'$ an angle equal to ABC; and a ray $C'Y$ exists in the same plane as the angle $C'B'X$, on the same side of $B'C'$ as $B'X$, and such that the angle $B'C'Y$ is equal to the angle BCA.

Then it is assumed (i) that the rays $B'X$ and $C'Y$ will meet at some point A' on the *same* side of $B'C'$ as that on which the angles $C'B'X$ and $B'C'Y$ are situated, and (ii) that the angle $B'A'C'$ will be equal to the angle BAC.

From this Postulate there follow immediately the two corollaries:

COROLLARY (i). If two angles of one triangle be respectively equal to two angles of another triangle, then their third angles are equal.

COROLLARY (ii). If the straight lines AX, BY lying in the same plane meet the straight line ABC so as to make the angles CAX, CBY equal to one another, then the straight line joining C to any point Q on BY will if produced meet AX in some point P, and the angles CQB, CPA will be equal.

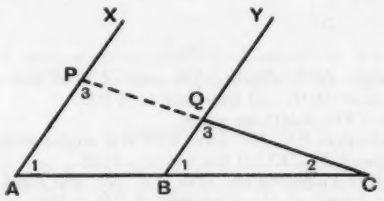


FIG. 2.

For if on CA a triangle be constructed having an angle at C equal to BCQ, and an angle at A equal to CBQ, its sides will be situated on the straight lines CA, AX and CQ. Hence by Wallis's Postulate AX meets CQ in some point P, and the angles CQB, CPA will be equal.

PROPOSITION I. (Euc. I. 27.)

Let the straight line ABCD meet the straight lines EBF, GCH in the same plane, so as to make the alternate angles EBC, BCH equal to one another; to prove that the straight lines EBF, GCH are parallel.*

If possible let the rays BF, CH meet at J.

On the ray BE take any point K; and on the ray CJ take any point M.

Join BM, CK.

In general the angles BCK, CBM will not be equal.

If BCK be greater than CBM, a straight line through C exists, which makes with CB an angle equal to CBM. It meets BK at some point K' between B and K. In this case CK' will be substituted for CK, and will be called CK.

If, however, the angle BCK be less than CBM, then a straight line through B exists which makes with BC an angle equal to BCK and meets CM at some point M' between C and M.

In this case BM' will be substituted for BM and called BM.

Thus straight lines BM, CK always exist which make the angles BCK, CBM equal.

Then the triangles BCK, CBM have two angles of the one BCK, CBK equal to CBM, BCM respectively of the other.

* Similar reasoning holds if FBC, GCB be taken as alternate angles.

Hence by Cor. (i) the remaining angles BKC, BMC are equal.
Since by hypothesis EBF, GCH meet at J, the triangle JCK exists.

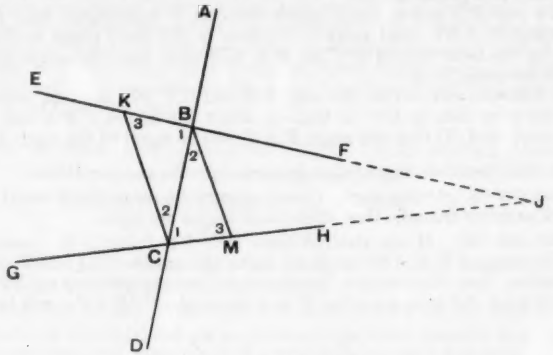


FIG. 3.

Further, the angle JCK, which is the sum of JCB and BCK, is equal to the sum of KBC and CBM, and therefore to KBM.

Also the angles JKC, BMC are equal.

Consequently the rays BK, MC make with BM angles which are respectively equal to the angles JCK, JKC of the triangle JCK.

Hence by Wallis's Postulate the rays BK, MC will meet at some point N (not shown in the figure) on the same side of BM as that on which the angles KBM, BMC are situated, and therefore on the ray BK.

Thus the straight lines BK, MC meet in two points J and N which are distinct, since J is on the ray BF, and N is on the ray BK, the production of BF through B. This is impossible.

Therefore EBF, GCH are parallel.

PROPOSITION II. (Euc. I. 28, Part I.)

If the straight lines a, b in the same plane meet the straight line $CABM$ at B, A respectively so as to make the exterior angle CAb equal to the interior and opposite angle CBa, then a, b cannot meet.*

Take any points D, F on b on opposite sides of A and join them to C.

Then, since the angles CAb, CBa are equal, it follows from Cor. (ii) that CD produced will meet a in some point E, and that the angles CDA, CEB will be equal.

Further, since the angles last named are equal, it follows from Cor. (ii) that CF produced will meet a in some point G, and that the angles CFA, CGB will be equal.

It will now be proved that a, b cannot meet.

Firstly, suppose that they could meet on the *same* side of CAB as that on which the equal angles CAb, CBa are situated. If so, let them meet at some point Z, as in the auxiliary figure. Join CZ. Then the angles CZA, CZB which correspond to CDA, CEB should be equal. This is impossible. [If Z existed it would be a point where D, E coincide.†]

* Similar reasoning holds if the exterior and corresponding interior angles be chosen in any of the other three possible ways.

† The proof of this case was, I believe, first given by Prof. T. P. Nunn, in the *Gazette*, vol. xi, p. 71 (May, 1922).

Secondly, a, b cannot meet on the side of CAB *opposite* to that on which the equal angles CAb, CBa are situated.

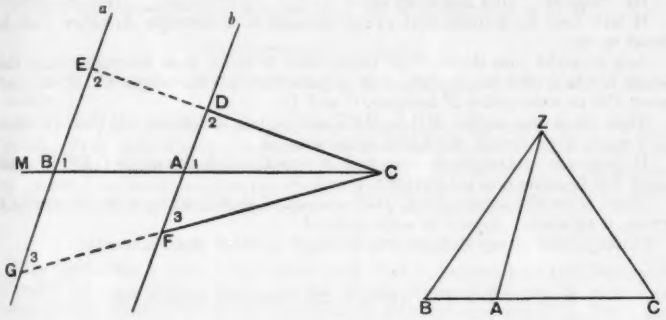


FIG. 4.

For if they did meet at some point Z, Z would be a point where F, G coincide; and then CZA, CZB should be equal, which is impossible.

PROPOSITION III.

To prove that if a be a straight line in a plane, and A a point in the plane, but not on a, then it is possible to draw only one straight line through A in the plane, which does not meet a.

Taking the figure of Prop. II., the point A is joined to any point B on a. BA is produced through A to any point C. Then A, B, and any other point on a determine a plane. In this plane a straight line b exists which

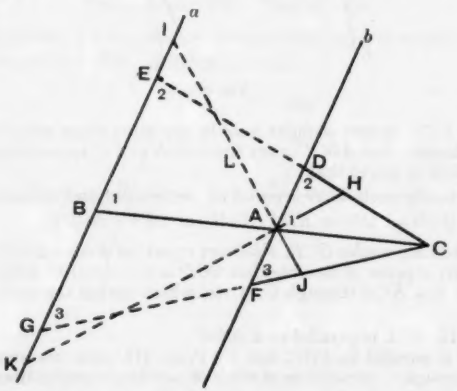


FIG. 5.

makes with AC an angle equal to CBa. D and F are any points on b on opposite sides of A; and, as proved in Prop. II., CD, CF meet a in E, G respectively.

Thus there exists a straight line b, passing through A, which makes with BAC an angle equal to that which a makes with BAC.

By Prop. II. b does not meet a.

It will now be proved that every straight line through A, other than b, must meet a.

Any straight line through A, other than b, must pass through either the angle CAD or the angle CAF. If it pass through the angle CAD, it must meet CD in some point H between C and D.

Then since the angles HDA, HEB are equal, it follows by Cor. (ii) that HA must, if produced, meet a in some point K.

If, however, a straight line through A pass through the angle CAF, it must meet CF in some point J between C and F.

Then, since the angles JFA, JGB are equal, it follows by Cor. (ii) that JA must, if produced, meet a in some point I.

Consequently every straight line through A, other than b, meets a.

PROPOSITION IV. (Euc. I. 29, First and Second Parts.)

If there are two straight lines in the same plane which do not meet, then any transversal meets them so as to make (i) *the exterior angle equal to the interior and opposite; and also* (ii) *the alternate angles equal.*

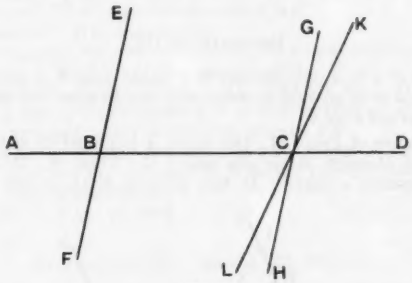


FIG. 6.

Let EBF, GCH be two straight lines in the same plane which do not meet.

Let any straight line $ABCD$ meet them at B and C respectively.

It is required to prove that

(i) the exterior angle GCD is equal to the interior and opposite angle EBD [or $HCD = FBD$, or $EBA = GCB$, or $ABF = BCH$];

(ii) the alternate angles GCB, FBD are equal [or $EBC = BCH$].

To prove (i) suppose if possible that GCD is not equal to EBD.

A straight line KCL through C exists, which makes the angle KCD equal to EBD.

By Prop. II. KCL is parallel to EBF.

But GCH is parallel to EBF, and by Prop. III. only one straight line can be drawn through C parallel to EBF. Hence the straight lines KCL, GCH coincide.

Therefore the angle GCD is equal to KCD, and therefore to EBD.

To prove (ii) suppose if possible that the angle GCB is not equal to FBD.

Then a straight line KCL through C exists, which makes the angle KCB equal to FBD.

By Prop. I. *KCL* is parallel to *EBF*.
But *GCH* is parallel to *EBF*.
Hence by Prop. III. the straight lines *KCL* and *GCH* coincide.
Therefore the angle *GCB* is equal to *KCB*, and therefore to *FBD*.

PROPOSITION V. (Euc. I. 30.)

Straight lines which are parallel to the same straight line are parallel to one another.

For let *a*, *b*, *c* be three straight lines in a plane, and let *b* and *c* be each parallel to *a*, then *b* must be parallel to *c*. For if *b* and *c* could meet, then through their point of intersection, two straight lines could be drawn parallel to *a*, but this is impossible by Prop. III. Therefore *b* and *c* do not meet, and they lie in the same plane, therefore they are parallel.

The preceding propositions contain all that is essential to the Theory of Parallel Lines. Before, however, the above propositions can be compared with Euclid's statement of his Postulate of Parallels and the 27th, 28th, and 29th Propositions of the First Book, it is necessary to introduce the right angle, but it should be noted that this introduction of the Right Angle only enables the properties of parallel lines already proved to be stated in a different form. *No new property of parallel lines is thereby obtained.*

Euclid defines a right angle thus: "When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle," and Euclid assumes as his Fourth Postulate that all right angles are equal.

It will be proved, however, that this postulate can be deduced as a consequence from the preceding theory of parallel lines.

Some other propositions will be required to be obtained from that theory first.

PROPOSITION VI. (Euc. I. 15.)

If two straight lines cut one another, the vertically opposite angles are equal.

Let the straight lines *ABC*, *DBE* meet at *B*.

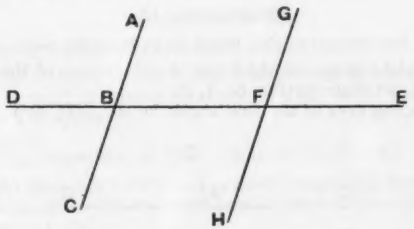


FIG. 7.

To prove that the angles *ABD*, *CBE* are equal.

A straight line *GFH* exists which passes through any point *F* on *BE* and makes the angle *GFB* equal to *ABD*.

Then, by Prop. II., *GFH* is parallel to *ABC*.

Then, by Prop. IV. (ii), *GFB* is equal to *FBC*.

Therefore *ABD* is equal to *FBC*, i.e. to *CBE*.

PROPOSITION VII.

To prove that all the angles of a triangle are together equal to a straight angle.

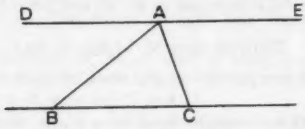


FIG. 8.

Let ABC be a triangle.

Then a ray AE exists which makes the angle CAE equal to ACB.

Produce EA to D.

Then DAE is parallel to BC by Prop. I.

Then DAB is equal to ABC by Prop. IV. (ii).

Hence $$ABC+BAC+ACB$$
$$=DAB+BAC+CAE$$

=the straight angle whose vertex is A which is bounded by the rays AD, AE.

PROPOSITION VIII.

To prove that two straight angles, whose vertices lie on a straight line, on which their bounding rays are situated, are equal.

In the figure of Prop. II. consider the straight angle whose vertex is at A with bounding rays AC, AB; and the straight angle whose vertex is at B with bounding rays BA, BM.

By the construction, the angle CAD is equal to ABE.

Then since AD is parallel to BE, it follows from Prop. IV. (i) that the exterior angle MBE is equal to the interior and opposite BAD.

Therefore $CAD+BAD=ABE+MBE$.

Therefore the two straight angles, viz. that with vertex A and bounding rays AC, AB, and that with vertex B and bounding rays BA, BM, are equal.

If we consider the straight angle as made up of the angles CAF, BAF, then it follows by Prop. VI. that these are equal to BAD, CAD respectively, and so form a straight angle equal to that made by BAD, CAD on the opposite side of the straight line CAB.

PROPOSITION IX.

To prove that two straight angles, which lie in the same plane, are equal.

(i) If the straight lines, on which the boundary rays of the straight angles lie, are parallel, let their vertices be A, B.

Let the bounding rays of the two angles be AC, AD, and BE, BF respectively.

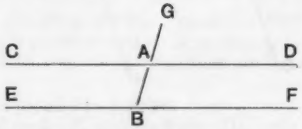


FIG. 9.

Join BA and produce to G.

Then the straight angle whose vertex is at A is equal to $CAG+GAD$
$$=EBA+ABF$$
=straight angle whose vertex is at B.

(ii) If the straight lines on which the bounding rays of the two straight angles lie are not parallel, let them meet at O.

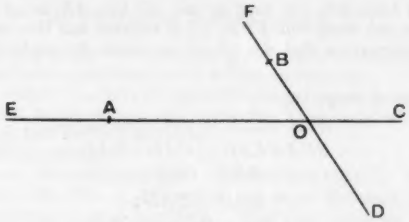


FIG. 10.

Then by Prop. VIII. the straight angle whose vertex is at A with bounding rays AC, AE is equal to the straight angle whose vertex is at O with bounding rays OC, OA, and is therefore equal to $AOB+BOC=COD+BOC$, since vertically opposite angles are equal (Prop. VI.).

Therefore it is equal to the straight angle whose vertex is at O with bounding rays along OD, OB, which by Prop. VIII. is equal to the straight angle whose vertex is at B with bounding rays along BO, BF.

PROPOSITION X.

To prove that straight angles in two different *planes are equal.*

If ABC be a triangle in one plane, then by Wallis's Postulate a triangle $A'B'C'$ exists in the other plane such that the angles of the first triangle are respectively equal to those of the second triangle.

Hence the sum of the angles of the triangle ABC is equal to the sum of the angles of the triangle $A'B'C'$.

But by Prop. VII. the sum of the angles of the triangle ABC is equal to a straight angle in the plane ABC, and the sum of the angles of the triangle $A'B'C'$ is equal to a straight angle in the plane $A'B'C'$.

Hence a straight angle in the plane ABC is equal to a straight angle in the plane $A'B'C'$.

Therefore all straight angles are equal.

PROPOSITION XI.

To prove that all right angles are equal.

Taking Euclid's Definition of a Right Angle and assuming that "Halves of Equals are Equal," it follows from the equality of all straight angles that all right angles are equal.

PROPOSITION XII. (Euc. I. 28, Part 2.)

If two straight lines ABC, GFH meet a third straight line $DBFE$ so as to make the two interior angles on the same side together equal to two right angles, then the straight lines are parallel.

Taking the figure of Prop. VI., it is given that

$$\begin{aligned} ABF+BFG &= \text{two right angles} \\ &= \text{a straight angle} \\ &= ABF+ABD\,; \end{aligned}$$

$$\therefore\ ABD=BFG,$$

i.e. the exterior angle is equal to the interior and opposite.

Hence by Prop. II. ABC, GFH are parallel.

PROPOSITION XIII. (Euclid's Postulate of Parallels.)

If two straight lines EB, LA meet the straight line AB, so as to make the two interior angles on the same side EBA, LAB together less than two right angles, then EB, LA will meet on that side of AB on which the angles EBA, LAB are situated.

Take the figure of Prop. III.

Since $EBA+LAB <$ two right angles;

$\therefore EBA+LAB < CAD+BAD.$

But $EBA=CAD;$

$\therefore LAB < BAD;$

$\therefore AL$ lies in the angle $BAD.$

But only one parallel to EB can be drawn through A, viz. $AD.$
Hence AL is not parallel to EB and therefore meets it.

PROPOSITION XIV. (Euc. I. 29, Part 3.)

If two parallel straight lines meet a third straight line, then they make the two interior angles on the same side together equal to two right angles.

Taking the figure of Prop. IV. (i) it was proved that the parallel lines EBF, GCH meet $ABCD$ so as to make $ABE=BCG.$

But $ABE+EBC=$a straight angle$=$two right angles;

$\therefore BCG+EBC=$two right angles.

Thus, instead of using the Postulate that all right angles are equal in constructing a theory of parallels, it has been shown that a theory of parallels when based on Wallis's Postulate suffices to prove that all right angles, as defined by Euclid, are equal.

Whenever a hypothesis invented to explain one set of facts is found to explain another set of facts for the explanation of which it was not invented, the validity of the hypothesis is confirmed.

This is what has happened in this case, and shows the great value of the idea contained in Wallis's Postulate.

CONCLUSION.

Had there been time I would have liked to include in this address an account of some other subjects which are not usually sufficiently explained, such as the Theory of Partial Fractions, the Convergency of an Infinite Geometric Series, when discussed in elementary treatises, and especially and above all, Euclid's renowned definition of Proportion; but these must stand over for some other occasion.

Those subjects of which I have spoken to you to-day, and others of a kindred nature, have been my companions for many years past. I have striven whenever I was teaching to make the expression of my meaning clearer as the years rolled by; to get back, as I have already said, to first principles whenever possible, in order that my pupils might realise that I was appealing directly to their reason rather than to their memory. To me this effort to attain the light of truth has been a source of pleasure and delight. If I have succeeded in conveying to you some of the fascination which the study of Mathematics has exerted over my mind and on my life, I shall be well content.

M. J. M. HILL.

THE GRAPHIC SOLUTION OF QUADRATIC EQUATIONS.

By Professor Alfred Lodge, M.A.

I want to urge on the Association the importance of including the geometrical solution of quadratic equations in the school curriculum.

We admit a certain amount of algebraic reasoning as it is, and there seems no reason against its extension, and there are many problems which admit of ready solution if this method is known.

It is wonderfully simple.

Let $x^2 - bx + ac = 0$ be the equation. Draw a line $OB = b$. At its ends draw, at right angles to it, $OA = a$, $BC = c$, in the same direction if ac is positive, in opposite directions if ac is negative. Then the circle on AC as diameter will cut OB in two points P, Q such that OP, OQ are the roots of the equation.

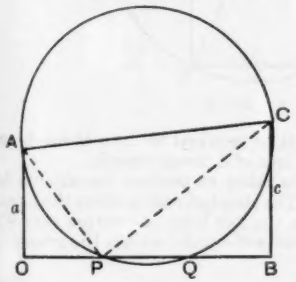


Fig. 1a.

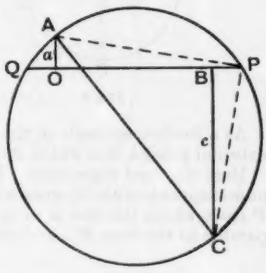


Fig. 1b.

For, the $\triangle$s AOP, PBC are similar, the angles being equal in that order;

$$\therefore \frac{AO}{OP} = \frac{PB}{BC}, \text{ i.e. } OP \cdot PB = ac,$$

so that, if OP is denoted by x, $x(b - x) = ac$, in Fig. 1a,

and $x(x - b) = -ac$, in Fig. 1b.

If b is negative, OB should be drawn to the left instead of the right.

In every case the roots are given in sign as well as magnitude.

It is not absolutely necessary to have OA, BC perpendicular to OB; they may be equally inclined to it, in the isosceles triangle sense; but in this general case AC is a chord and not a diameter. The essential matter is to have the centre of the circle on the perpendicular bisector of OB, and to have AC as a chord. However, it is usually best to make AOB and CBO right angles.

We already have two cases in Euclid, viz. (i) $x^2 - ac = 0$ (Fig. 2), and (ii) the golden section (Fig. 3), where a line of length a is to be cut into two parts x, $a - x$ such that

$$x^2 = a(a - x),$$

$$\text{i.e. } x^2 + ax - a^2 = 0.$$

In this case OB should be drawn to the left, and it is easy to see that its mid-point is the centre of the circle, and that we have practically Euclid's construction: OP is equal to x.

The swinging round of the radius MA into the position MP automatically cuts MO from it, leaving $OP = x$. The modern way of using only the $\triangle MOA$

and cutting $MX = MO$ from the hypotenuse MA is clearly another way of isolating the required length x.

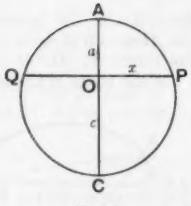


FIG. 2.

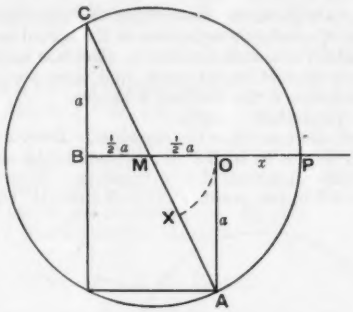


FIG. 3.

As a further example of the method, let it be required to draw through an external point a line which shall bisect the area of a given triangle.

Here we need some care. If the point is taken at random the algebra is unmanageable without some adjustment. The simplest case is when the point P from which the line is to be drawn is on the line from one vertex (say A) parallel to the base BC, and the bisecting line will cut BC within the triangle.

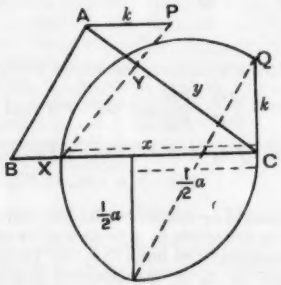


FIG. 4.

Let it cut AC at a distance y from C, and BC at a distance x from C, so that $2xy = ab$, and let $AP = k$.

Then, by similar triangles,

$$\frac{k}{x} = \frac{b-y}{y} = \frac{b}{y} - 1 = \frac{2x}{a} - 1;$$

$$\therefore\ 2x^2 - ax = ak,$$

$$i.e.\quad x^2 - \tfrac{1}{2}a \,.\, x = \tfrac{1}{2}a \,.\, k;$$

therefore all that has to be done is to draw from the mid-point of BC a line of length $\frac{1}{2}a$ to a point R, and from C a line $CQ = k$, in opposite directions,

and describe a circle having QR as diameter, cutting BC in X. Join PX, cutting AC in Y.

The triangle CXY will be half ABC.

For all other positions of P we must make a triangle of the same area as ABC (taking care not to alter the positions of the two lines that PXY is to cut) so that the line joining P to a vertex of the new triangle will be parallel to the base, and complete the above construction with the new triangle.

The annexed figure shows the construction when P is on AB produced.

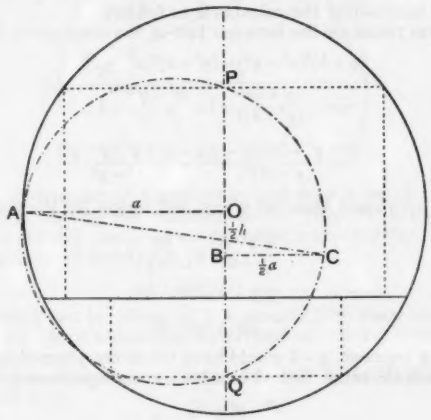


FIG. 5.

If we require CXY to be one m^{th} of ABC, the equation is $x^2 - \frac{a}{m} \cdot x = \frac{a}{m} \cdot k$, and is equally simple, if we are sure beforehand which sides of the triangle will be cut.

Another problem which leads to a quadratic is to construct a rectangle of maximum area in a segment of a circle. Consider the segment to be a major one.

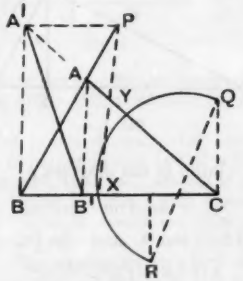


FIG. 6.

If O is the centre, h the distance of the chord of the segment from O, and $y+h$ the height of the rectangle, we require $(y+h)\sqrt{a^2-y^2}$ to be a maximum, where a is the radius of the circle.

Either by the calculus, or otherwise, this leads to the equation

$$y^2+\tfrac{1}{2}h\,.\,y=\tfrac{1}{2}a^2.$$

If the segment is a minor one, the sign of h must be changed, and the equation is

$$y^2-\tfrac{1}{2}h\,.\,y=\tfrac{1}{2}a^2.$$

It will be found, on drawing the circle indicated by either of these equations that the same circle gives the maximum rectangles in both segments.

One method of avoiding the calculus is as follows:

The maximum rectangle lies between two of the same area; therefore let

$$(y+h)^2(a^2-y^2)=(y'+h)^2(a^2-y'^2),$$

$$i.e.\quad \frac{(y+h)^2}{(y'+h)^2}-1=\frac{a^2-y'^2}{a^2-y^2}-1;$$

$$\therefore\ \frac{(y+y'+2h)(y-y')}{(y'+h)^2}=\frac{(y+y')(y-y')}{a^2-y^2}.$$

Now, cancel $y-y'$, and then put $y'=y$, and we get for the maximum case

$$\frac{2(y+h)}{(y+h)^2}=\frac{2y}{a^2-y^2};$$

$$\therefore\ y(y+h)=a^2-y^2;$$

$$\therefore\ 2y^2+hy=a^2.$$

For the minor segment, $y-h$ would have taken the place of $y+h$.

We can also use the same method to solve certain equations in two variables. Suppose

$$x^2-y^2=a^2,$$

$$xy=b^2.$$

It is easy to arrange for the first equation, by taking a right-angled triangle with base $OB=a$; then x will be the hypotenuse, and y the other side. We now have to arrange for $xy=b^2$.

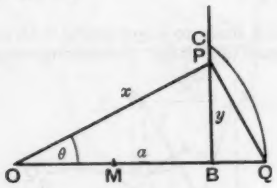


FIG. 7.

It will be noted that $\frac{y}{a}=\tan\theta$ in the diagram;

$$\therefore\ x\tan\theta \text{ must} = b^2/a.$$

Let PQ be $x\tan\theta$ and let $OQ=z$.

Then $$z(z-a)=PQ^2=(b^2/a)^2.$$

The construction circle required here will have the mid-point M of OA for its centre, and MR for its radius if we make $AR=b^2/a$.

This circle will cut OA, produced, in the required point Q, and then P is found by making $PQ=b^2/a$.

Another pair of equations is

$$x^2+y^2=a^2,$$
$$xy=b^2.$$

This is simpler.

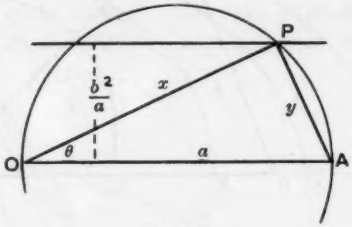


FIG. 8.

Take $OA=a$ as diameter of a semicircle, and draw a parallel line at a distance b^2/a from it, cutting the semicircle in P. Then OP, AP are the values of x and y. For, in this case $y=a\sin\theta$; hence $x\sin\theta=b^2/a$.

These are solutions of the equations

$$x^4\pm a^2x^2\mp b^4=0.$$

A problem which can be solved in this manner is to draw through a point O within a circle a chord which shall be divided at O into parts x, y such that x^2 shall equal $y(x+y)$.

In this case xy is constant, equal to b^2, where $2b$ is the length of the chord which is bisected at O.

$$\therefore\ x^2-y^2=b^2.$$

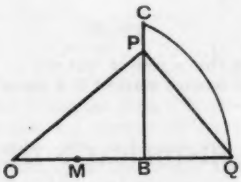


FIG. 9.

Hence, let OB, BC be two sides of a square of length b. Bisect OB in M, and produce it to Q so that $MQ=MC$, and make $PQ=b$.

Then $OP=x$, $PB=y$.

The circle in which O lies is not shown. Any circle through B with its centre on the line through O perpendicular to OB will do, provided that it is large enough—provided, in fact, that its diameter is not less than $OP+PB$.

One more diagram (Fig. 10) is, I think, worth inserting. It shows how to draw, from P, lines which will enclose with the lines AC, BC triangles whose areas are respectively 1, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, the area of ABC being taken as unity and AP being parallel to BC. The quadratics on which the construction depends are of the form $my^2-ay=ak$, where $1/m=1$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ respectively.

It should be noticed that the only really needful line besides CQ is the line of centres, which starts at the midpoint of CQ, cuts BC at 45°, and is half as long as the parallel line through C. The centres divide it into parts corresponding to the values of $1/m$, and all the circles go through Q. The other

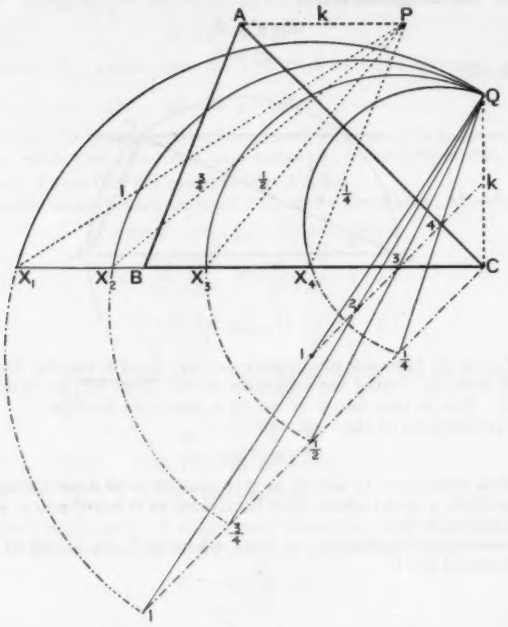


FIG. 10.

lines may be required for the *argument*, but not for the construction. It is only by accident that the centres marked 3, 4 lie on sides of the triangle.

ALFRED LODGE.

HISTORICAL NOTE

ON THE GRAPHIC SOLUTION OF QUADRATIC EQUATIONS.

BY THE REV. J. J. MILNE, M.A.

Descartes at the beginning of his work on Geometry (1637), which would now be called a treatise on Algebraical Geometry, in order to shew the relation between Algebra and Geometry considers four types of quadratic equations. It will be noticed that these are special forms, the constant term of a quadratic equation being as a rule, not a square, but the product of two lengths, and necessitating the use of Euc. VI, 13 to find the equivalent square.

Prof. Lodge's method by the construction of a circle gives us at once in one figure the solution of the quadratic in its general form. I may mention that this method was discovered by him in 1907, and in the *Gazette* for April 1909 (vol. v. p. 81) was employed by him in finding the common points and pairs of corresponding points on two coaxial ranges. Previous to this discovery the usual method of finding them was to project the ranges on to a circle or conic and to make use of the Pascal line (see my *Cross Ratio Geometry*, Ch. VII.).

The types of quadratics considered by Descartes are:

(1) $x^2 - ax = b^2$, $\quad x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 + b^2)}$;

(2) $x^2 + ax = b^2$, $\quad x = -\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 + b^2)}$;

(3) $x^2 - ax = -b^2$, $\quad x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - b^2)}$;

(4) $x^2 + ax = -b^2$, $\quad x = -\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - b^2)}$.

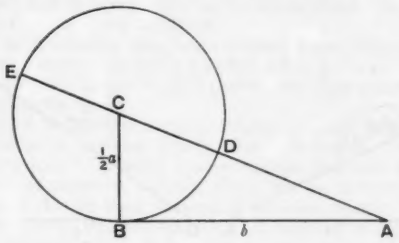


FIG. 1.

To solve (1) and (2) he drew a line $AB = b$, and erected a perpendicular $BC = \frac{1}{2}a$. With centre C and radius CB, he described a circle, and he drew AC cutting the circle in D and E. Then since $AC = \sqrt{(\frac{1}{4}a^2 + b^2)}$, the positive roots of (1) and (2) are:

(1) $AE = CE + AC = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 + b^2)}$,

$[\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 + b^2)} = DC - AC = -AD]$;

(2) $AD = -DC + AC = -\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 + b^2)}$,

$[-\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 + b^2)} = -CE - AC = -AE]$;

so that the solutions of (1) are AE, $-AD$; and of (2) are AD, $-AE$.

It should be noticed that Descartes gives only the positive roots.

To solve (3) Descartes adopts a slightly different construction. He drew $AB = b$, $BC = \frac{1}{2}a$, and described the circle as before, and through A drew a line perpendicular to AB cutting the circle in D and E.

Then $DH = HE = \sqrt{(CD^2 - CH^2)} = \sqrt{(\frac{1}{4}a^2 - b^2)}$.

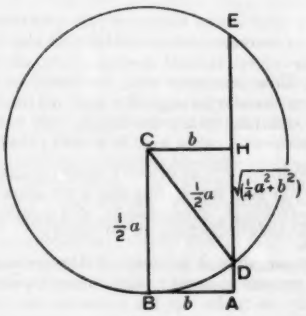


FIG. 2.

To solve (3), $AE = AH + HE = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - b^2)}$,

$AD = AH - DH = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - b^2)}$.

Descartes does not give a solution of (4), in which both roots are negative.

Maclaurin in his *Algebra* (1771) solves the equations as follows:

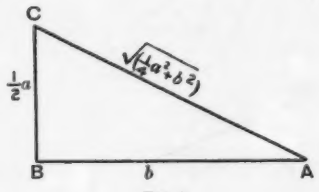


FIG. 3.

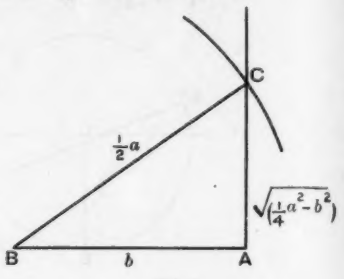


FIG. 4.

In Fig. 3,

(1) $BC \pm CA = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 + b^2)}$; (2) $-BC \pm CA = -\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 + b^2)}$.

In Fig. 4,

(3) $BC \pm CA = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - b^2)}$; (4) $-BC \pm CA = -\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - b^2)}$.

It will be seen that neither Descartes nor Maclaurin gives a method applicable to a constant term which is not a square. The general method given by Prof. Lodge covers all cases. JOHN J. MILNE.

GLEANINGS FAR AND NEAR.

419. The Anglo-Dane appears to possess an aptitude for mathematics which is not shared by the native of any other English district as a whole, and it is in the exact sciences that the Anglo-Dane triumphs. Newton is the supreme figure of Anglo-Danish science.—Havelock Ellis, *A Study of British Genius* (1904).

420. Newton, along with views the most just and important concerning the nature and methods of science, had something of the tendency, prevalent in his time, to suspect or reject, at least speculatively, all elements of knowledge except observation. This tendency was, however, in him so corrected and restrained by his own wonderful sagacity and mathematical habits, that it scarcely led to any opinion which we might not safely adopt.—Whewell, *Philosophy of the Inductive Sciences*, vol. ii. p. 295 (1847).

421. The French philosophers at present chiefly follow Malebranche. They admire Sir Isaac Newton very much, but don't yet allow of his great principle: it is his particular reasonings, experiments, and penetration, for which they so much admire.—Spence's *Anecdotes*, p. 29 (1858).

422. Sir Isaac Newton, though so deep in Algebra and Fluxions, could not readily make up a common account: and, when he was Master of the Mint, used to get somebody to make up his accounts for him.—(Pope) Spence's *Anecdotes*, p. 132 (1858).

GEOMETRICAL CONGRUENCE.

By W. C. Fletcher, M.A.

Many teachers know that Euclid's proofs by superposition have been much criticised, but to most of us, perhaps, even after the publication of the *Report on Geometry*,* the matter has remained somewhat obscure. As for purposes of my own I have had to try to clear my mind on the subject, possibly what I have so learnt may be of use to others.

The doctrine of Relativity may have created some difficulties for us, but in this particular matter it is really helpful, for it gives us a starting-point. Conversely, the line of thought I shall follow may help some over the initial difficulties of Relativity itself.

If the Theory of Relativity is true, congruence does not generally exist. The first thing is to understand clearly what this means. We are of course familiar with the idea that solid bodies change size and shape under the influence of heat, of pressure, and the like; but we are now told that they also suffer changes due to their changes of position relative to other bodies. Similarly, we have long been familiar with the bending of light due to change in the medium through which it is passing; we are now told that even in vacuo its path is affected by neighbouring matter.

The consequence is that we can neither assume at the outset the existence of absolutely rigid bodies, nor fall back on the path of a ray of light as a natural 'straight line'.

Alternatively we can say that space—the actual space in which we locate our experiences—is no longer mere emptiness, but has properties which control the behaviour of material bodies and of light.

Next it is essential to realise that bodies may change size and shape, but the changes may be so cunningly coordinated that they are not betrayed by failure of congruence; on the other hand, of course, they may be so betrayed. To take a crude but useful example: imagine a wheel consisting of rim and spokes all of the same material; if it is heated everything expands, but this may well take place in such a fashion that the wheel does not warp or buckle; if we use measuring rods of the same material, heated with the wheel, the actual measurements will remain unaltered too. If, on the other hand, the spokes are of one material and the rim of another with a smaller coefficient of expansion, the wheel may buckle, and change may therefore be betrayed without need for measurement.

Relativity asserts that the changes in material figures due to the qualities of space in the neighbourhood of matter are of the latter kind; the ratio of the circumference of a circle to its radius depends on both position and orientation with regard to, say, the earth or the sun. This again is only a crude statement; more properly the ideal geometrical circle does not exist at all.

For a circle is not a mere line, everywhere equidistant from a centre; it is a *plane figure* enclosed by such a line, and, according to Relativity, planes do not generally exist in the neighbourhood of gravitating matter.

A boy has no difficulty with the idea of a plane—the thing is so familiar in experience; but he often has considerable difficulty in grasping the *definition* of a plane: "a surface such that, any two points therein being taken, the straight line joining them lies wholly in the surface." And the definition presupposes an axiom, not expressed by Euclid, but supplied by more modern geometers, that such surfaces exist. And this axiom lies at the base not only of the familiar Euclidean geometry, but of the more general geometry which Professor Baker has been recommending to us.

* *The Teaching of Geometry in Schools*: M.A. Report, 2nd ed. 1925.

Once more, it results from Relativity that, generally speaking, surfaces having this property of planes do not exist. It is essential to realise exactly what this means.

There is a preliminary question about the 'straight line' itself. In pure theoretical geometry this is defined by axioms, but in the physical geometry with which Relativity is concerned we are not free to ignore the qualities of space, and the unique line joining two points can no longer be defined *à priori*, but requires investigation. The place of the ideal 'straight line' defined by axioms is taken by the geodesic, that is, the shortest line joining the two points.

We are, of course, familiar with the idea of geodesics on a curved surface: the constraint of the surface prevents a piece of string taking the actual shortest track from point to point, and the geodesic is simply its track while confined to the surface. But in free space it is less easy to realise how the 'shortest' track can be other than straight. A simple way is to imagine the space unequally heated; then it is easy to realise that, say, 99 links of a chain laid on a curve passing partly through hotter regions might suffice to connect two points, while 100 might be required along the straight line. Thus the form of the geodesic depends on the qualities of the space.

Now consider a framework of three geodesics, forming the outline of a triangle ABC; do they form a triangle at all? They do so only if there is a unique surface filling the framework and having the property ascribed to the plane. Connect A to points in BC by geodesics: these form a surface of some kind. Connect B to points in CA, and C to points in AB. Are the three surfaces identical? Clearly this will depend on the shape of the geodesics and cannot be assumed. If they are not, there is no unique surface filling the outline, and the plane, and therefore the triangle, does not exist: three lines of given length no longer define a definite figure.

Even if there is such a unique surface, does congruence follow without a further condition? It seems obvious from the consideration of the geometry of a surface that it does not. Two triangles can be drawn in different places on the surface with their corresponding sides equal; if the surface is a plane or a sphere, the triangles will be congruent, *i.e.* their corresponding angles, and all other related lines, medians, perpendiculars, etc., will be equal; but if the surface is an ellipsoid, this will not in general be true.

But the two-dimensional case is here misleading. It is a very remarkable fact that if, in three dimensions, the conditions for planes are satisfied, the further conditions, which at first sight appear requisite for congruence, follow identically.

Relativity, as already stated, asserts that the qualities of actual space are such that in general there are neither planes nor congruent figures; but we may reasonably set out to enquire what the properties of space would have to be to render these things possible.

Indirectly this was done long ago, for both Euclid and Lobatscheffski, proceeding on the basis of axioms, suggested indeed by experience, but taken as absolute and so independent of experience, worked out systems of geometry; their axioms included the existence of planes and of congruence, and it is possible to express their assumptions in terms of the properties of space; so we know certain sets of properties consistent with congruence. Riemann added a third system, discarding the axiom that two straight lines only meet once, and this system again can be expressed in terms of the properties of space. But by a more general method, due also to Riemann, the problem can be investigated from the other end, and the question asked and answered, what must be the properties of space if congruence is to exist.

An initial assumption is made, viz. that there is meaning in saying that an infinitesimal length in one place is equal to one in another. In the more

recent developments of Relativity this assumption is abandoned or at least modified, with important results; but we are not concerned with this here.

It is assumed then that an element of length may be expressed by the formula

$$ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + 2g_{23}dx_2dx_3 + 2g_{31}dx_3dx_1 + 2g_{12}dx_1dx_2,$$

where dx, etc., may be elements of Cartesian or other coordinates and the g's are functions of those coordinates.

This formula requires some consideration. We are familiar with the Euclidean formula $ds^2 = dx^2 + dy^2 + dz^2$. This clearly assumes that space is the same everywhere; but suppose, reverting to our illustration from heat, that one region is hotter than another, and that all measuring instruments contract and expand in the same way. And suppose first for simplicity that the expansion depends simply on the place, not on the direction in which the rod is laid; then if at the point x, y, z the length of the rod were $1/\sqrt{\mu}$ of its normal length, the element *as measured* would be given by

$$ds^2 = \mu(dx^2 + dy^2 + dz^2).$$

If we generalise our assumption so that the behaviour of the rod depends on orientation as well as position, we get the more complicated formula given above.

On this basis it can be proved that the conditions for the existence of congruence are $B_{\mu\nu\sigma\rho} = a(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})$ where a is constant throughout space. The B's are the components of the Riemann-Christoffel Tensor familiar to students of Relativity. In two dimensions there is only one of these equations; in three there are six; in four there are twenty. In two and in three dimensions, but not in four, they are equivalent to $G_{\mu\nu} = \beta g_{\mu\nu}$, in which form they are Einstein's equations for the problem of gravitation.

Applying these to the simple case $ds^2 = \mu(dx^2 + dy^2 + dz^2)$, we get six differential equations for μ, three of the type

$$\frac{1}{\mu}(\mu_{xx} + \tfrac{1}{2}\mu_{yy} + \tfrac{1}{2}\mu_{zz}) - \frac{1}{\mu^2}(\mu_x^2 + \tfrac{1}{4}\mu_y^2 + \tfrac{1}{4}\mu_z^2) = \beta\mu$$

and three of the type $$\mu_{yz} = \frac{3}{2\mu}\mu_y\mu_z,$$

where suffixes denote partial differentiations with respect to x, y, z.

The general solution * of these equations is

$$\mu = \left[\frac{C}{k^2 + (x-a)^2 + (y-b)^2 + (z-c)^2}\right]^2,$$

where C, a, b, c are arbitrary constants and $k^2 = 2/\beta$. The values of a, b, c cannot, of course, be determined from the differential equations, and as we are speculating and have no actual observations or boundary conditions from which to determine them, we can only take them as the coordinates of an unknown but *unique and definite point.*

If we take this point as origin, we may write the solution in the forms:

$$\mu = \text{constant (if } \beta \text{ is zero)};$$
$$\mu = A/r^4 \text{ (if } \beta \text{ is infinite)};$$
$$\mu = \left(\frac{2k^2}{k^2 \pm r^2}\right)^2.$$

Of these the first gives us merely the characterless space of Euclidean geometry. The second leads to the inverse of Euclidean space with regard to the origin; straight lines and planes become circles and spheres passing through the origin; with the variable scale the whole of Euclidean geometry remains true, and all we have got is a more complicated expression; some-

* There is another more special solution, but it does not affect the argument.

what as the reflection of space in a convex mirror, if interpreted with a suitable varying scale, gives a more complicated representation of Euclidean geometry.

The other solutions, $\left(\frac{2k^2}{k^2 \pm r^2}\right)^2$, give us spaces with quite different characters, and lead us to the hyperbolic geometry of Lobatscheffski and the spherical geometry of Riemann. In the former (denominator $k^2 - r^2$) measuring rods contract as we depart from the origin; space is bounded by the sphere $r = k$, but appears infinite to measurement; geodesics are circular arcs cutting the boundary orthogonally. In the latter (denominator $k^2 + r^2$) rods expand; there is no boundary, space embraces the whole of Euclidean space, but is finite to measurement; geodesics are complete circles cutting the sphere $r = k$ at the ends of a diameter.

In both cases suitable assemblages of geodesics form spheres: in the former, segments of spheres cutting the boundary orthogonally; in the latter, complete spheres cutting $r = k$ in a diametral plane. In each case these spheres, or segments of spheres, intersect one another in circles, or circular arcs, which are geodesics, *i.e.* these spheres have the essential property assumed in the definition of 'planes'.

In both cases congruence exists. Account has, of course, to be taken of the varying scale, and it turns out that congruence is recognisable by the fact that figures, one of which is the inverse of the other in any of the spheres which take the place of planes, are equal in all respects.

These spaces, which I call Poincaré's Models, have been more fully described in a previous number of the *Gazette*.* More complicated models can be constructed, but they lead to no different geometries—just as $\mu = A/r^4$ gives only a different representation of Euclidean space.

These three geometries then, or these three kinds of space, are the only ones which admit of congruence—and none of them is the geometry of space as revealed by Relativity. This only helps to make clearer their true character. It used to be asked, Which of them is true? meaning which of them corresponds with the actual facts of observation. The answer is that none of them is true in this sense. They are all alike pure structures of thought based on axioms; all true in the sense that they are perfectly self-consistent and free from contradiction. Nor, so far as is yet known, is either of them a closer representation of the facts than the others—with an important qualification.

Euclidean geometry is not only the simplest of the three, but the only one which is unique. The others depend on an assumed world-constant k, and until this is determined their formulae are useless, just as are the formulae of ordinary spherical geometry until the radius of the particular sphere is known to which they relate.

It is true that just as a Euclidean model of the other systems can be constructed, a model of the Euclidean system can be constructed in either of the others; but admitting that they are in this sense much on the same footing, it remains that the Euclidean space and geometry have the advantages of simplicity and uniqueness. It would therefore be reasonable to take these as the norm and to use them to frame certain definitions.

It used to be said that Euclid was the expression of the properties of rigid bodies. The question was inevitable, What is a rigid body? and it was unanswerable. But we can reverse the process and say: Euclid is a system of pure thought based on axioms, beyond attack from 'experiments'; a rigid body or an unchanging figure is one which conforms to the results of the Euclidean system. This means that an absolutely featureless space and absolutely unchanging measuring rods are to be defined as those conforming to Euclid.

* See vol. xi, p. 260: "A Method of Studying Non-euclidean Geometry."

The other two systems would then be those of spaces with certain characteristics, the result of which would be that bodies and figures expand or contract as they depart from a certain fixed point, the expansion or contraction following such a law that the changes are so coordinated as to be undiscoverable by observation: congruence persists although there is change.

The simplest illustration of the principle is perhaps this: imagine one 'straight line' and the locus of a point at a constant distance from it; is this locus another straight line? In Euclid it is; in the others it is not. It is not unreasonable to take Euclid's result as the definition of constant distance, and to say that, in the others, a distance called constant 'really' expands or contracts.

These three geometries then, or the kinds of space which they represent, are the only ones consistent with congruence; each of them relates to figures of finite dimensions whose properties are independent of position; in short, each of them has a trigonometry, or a method of calculating any element of a figure when sufficient elements are given, wherever the figure may be. The typical formulae of the three are:

$$\sin A : \sin B : \sin C = a : b : c;$$

$$\sin A : \sin B : \sin C = \sin\frac{a}{k} : \sin\frac{b}{k} : \sin\frac{c}{k};$$

$$\sin A : \sin B : \sin C = \sinh\frac{a}{k} : \sinh\frac{b}{k} : \sinh\frac{c}{k}.$$

In space with characteristics different from these there can be no 'integral' geometry, no geometry of finite figures independent of position; we can only proceed from point to point, and geometry remains 'differential'.

THE INTRODUCTION OF TIME.

But Relativity asserts that in all space-measurements time is necessarily entangled, and the basis on which we have been proceeding is therefore insufficient. The simplest formula analogous to $ds^2 = dx^2 + dy^2 + dz^2$ admitted by Relativity is $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.

This formula is valid in certain simple cases, but there is an element of ambiguity even then. If the formula is valid for one observer, it is also valid for a second observer moving uniformly with regard to the first; but whether it is valid for any remains obscure. Though it is contrary to relativistic principles, perhaps we had better assume that the formula belongs primarily to a hypothetical fixed space in which light moves uniformly, and secondarily to any space moving uniformly with regard to this. The dx, dy, dz are then primarily Cartesian elements in this hypothetical space, dt the time element proper to that space; but they may also be the corresponding elements as measured by an observer moving uniformly through the space—differing individually from the primary values, but giving the same value to ds^2.

Our present business, however, is to consider what modifications of this formula are consistent with congruence. Congruence in four dimensions is perhaps not very easy to understand, but at least, just as congruence in three dimensions includes that in two, so congruence in four includes that in three. We need not then begin *de novo*, but can start with the expression already known for the space terms and write

$$ds^2 = g_{44}dt^2 - \left(\frac{2k^2}{k^2 \pm r^2}\right)^2 (dx^2 + dy^2 + dz^2).$$

But there are in fact reasons for choosing the form with k^2+r^2, and it will be sufficient to start with that. We then have to determine g_{44} so that

$$ds^2=g_{44}dt^2-\left(\frac{2k^2}{k^2+r^2}\right)^2(dx^2+dy^2+dz^2)$$

satisfies the conditions for congruence, viz.

$$B_{\mu\nu\sigma\rho}=a(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho}).$$

If we assume that g_{44} depends on position in space only and is independent of the time, many of these twenty equations are satisfied identically; others are satisfied because we have already determined the coefficients of dx^2, dy^2, and dz^2; the rest lead to

$$g_{44}=\left(\frac{k_1^2-r_1^2}{k^2+r^2}\right)^2,$$

where r_1 is the distance from a second unknown point O_1, and $k_1^2=k^2+OO_1^2$.

As before, the differential equations themselves, without boundary conditions, do not enable us to determine O_1; all we can say about it is that it is a definite, though unknown, point. For the present it will be sufficient to identify it with O and to write the formula

$$ds^2=\left(\frac{k^2-r^2}{k^2+r^2}\right)^2dt^2-\left(\frac{2k^2}{k^2+r^2}\right)^2(dx^2+dy^2+dz^2)$$

where r is the distance from a definite point.

The space geodesics we know already; they are circles cutting $r=k$ at the ends of a diameter, and their geometry is spherical. But now we have to consider light tracks as well.

In constructing his Model, Poincaré, being free to follow his fancy and make a vivid picture, so arranged things that light tracks coincided with geodesics. To that end he had to choose a certain law for the index of refraction. Now the matter is no longer arbitrary: space-geodesics and light tracks have to be determined from the formula for ds^2.

To determine light tracks we have to put $ds^2=0$ and make $\int_A^B dt$ stationary. This requires

$$\delta\int_A^B\left(\frac{2k^2}{k^2-r^2}\right)\sqrt{dx^2+dy^2+dz^2}=0.$$

That is, the factor $\frac{2k^2}{k^2+r^2}$ which determines the space-geodesics is replaced by $\frac{2k^2}{k^2-r^2}$.

But this is the fundamental equation for hyperbolic space, and the light tracks are the geodesics of that space, viz. circular arcs cutting $r=k$ orthogonally.

We know already, or see from the formula, that measuring rods expand as they depart from the centre, by the factor $\frac{k^2+r^2}{2k^2}$; how do clocks behave?

Our coordinate elements, dt just as dx, dy, dz, are conceptual; in fact, we have imagined them as the elements as measured by a fixed observer. All actual measurements whether of time or space are determined by the formula. So if an observer is measuring time alone, the relation between his measured time, ds, and the conceptual time, dt, is $ds=\frac{k^2-r^2}{k^2+r^2}dt$.

That is to say, just as, relative to a conceptual constant measure of space defined by the properties of Euclidean geometry, rods expand, so, relative to a conceptual constant measure of time, clocks in this space run slow by the factor $\frac{k^2-r^2}{k^2+r^2}$ and stop altogether at $k=r$.

The velocity of light is given by $ds=0$, and is therefore $\frac{k^2-r^2}{2k^2}$. But this is in conceptual coordinates; remembering that the 'foot' lengthens by the factor $\frac{k^2+r^2}{2k^2}$ and the 'second' by the factor $\frac{k^2+r^2}{k^2-r^2}$, we see that the velocity of light *as measured* is constant, viz. unity.

It follows that observers in this space, treating light velocity as uniform, will want to measure distance by light-time. There are therefore two conflicting systems of measurement: that with which we started—expanding measuring rods used along space-geodesics; and that by light-time along light tracks. Which will they adopt?

Clearly that by light, for this alone is available for any but purely local purposes; for the small distances for which rods can be used there will be no difference.

Our system then is in a sense turned inside out: it began as an unbounded space giving the phenomena of spherical geometry; the introduction of time and light has isolated a definite, bounded part of it, viz. that inside the sphere $r=k$; the spatial geometry, based on light tracks and light time, has become hyperbolic; the space accessible by light, though bounded, appears infinite; clocks, and presumably all changes, slow down as the boundary is approached, and would stop if the boundary could be reached.

This extension of Poincaré's Model is what is known as 'de Sitter's Empty World.' An account of it is given by Professor Eddington in §§ 67-70 of *The Mathematical Theory of Relativity*; but I venture to think his account is confused in places, because he has in effect treated the arbitrary constants in the solution of the differential equations as if they were arbitrary in the sense that they could have several values at once. The view taken here is that this 'empty world' has one definite centre, has a definite form in space and a definite boundary at which light stops.

Our Euclidean formula for ds^2 can of course be expressed in 'internal' form, *i.e.* in terms of quantities as measured, and that in various ways.

First, as we started with the formula

$$\left(\frac{2k^2}{k^2+r^2}\right)^2(dx^2+dy^2+dz^2)$$

it is natural to use spherical geometry, *i.e.* that dependent on the expanding measuring rods. The formula then becomes

$$ds^2=\cos^2\rho/k\,.\,dt^2-\{d\rho^2+k^2\sin^2\rho/k\,.\,(d\theta^2+\sin^2\theta\,d\phi^2)\}.$$

But we saw that we were driven to abandon space-geodesics and replace them by light tracks; the suitable shape is then

$$ds^2=\frac{dt^2-\{d\rho^2+k^2\sinh^2\rho/k\,.\,(d\theta^2+\sin^2\theta\,d\phi^2)\}}{\cosh^2\rho/k}.$$

These formulae have here been obtained by transformation from the Euclidean formula; they can of course be determined *à priori* from the differential equations. For instance, we may start with

$$ds^2=\frac{dt^2-\{d\rho^2+k^2\sinh^2\rho/k\,.\,(d\theta^2+\sin^2\theta\,d\phi^2)\}}{\mu^2}$$

and determine μ and β so as to satisfy the equations $G_{\mu\nu}=\beta g_{\mu\nu}$.

We may assume from our previous work that μ is a function of the distance from a point, but we are not entitled to assume that this point is the origin, which is now arbitrary. We may either shift the origin to this unknown point, or, perhaps better, keep the origin arbitrary and assume that μ is a function of ρ and θ; ϕ we may ignore.

Writing l for ρ/k, the equations then prove to be:

$$\frac{3\mu_{ll}}{\mu}-\frac{3\mu_l^2}{\mu^2}+\frac{2\mu_l}{\mu}\frac{\cosh l}{\sinh l}-2+\frac{1}{\sinh^2 l}\left[\frac{\mu_{\theta\theta}}{\mu}-\frac{3\mu_\theta^2}{\mu^2}+\frac{\mu_\theta}{\mu}\frac{\cos\theta}{\sin\theta}\right]=\frac{k^2\beta}{\mu^2},$$

$$\frac{\mu_{ll}}{\mu}-\frac{3\mu_l^2}{\mu^2}+\frac{4\mu_l}{\mu}\frac{\cosh l}{\sinh l}-2+\frac{1}{\sinh^2 l}\left[\frac{3\mu_{\theta\theta}}{\mu}-\frac{3\mu_\theta^2}{\mu^2}+\frac{\mu_\theta}{\mu}\frac{\cos\theta}{\sin\theta}\right]=\frac{k^2\beta}{\mu^2},$$

$$\frac{\mu_{ll}}{\mu}-\frac{3\mu_l^2}{\mu^2}+\frac{2\mu_l}{\mu}\frac{\cosh l}{\sinh l}+\frac{1}{\sinh^2 l}\left[\frac{\mu_{\theta\theta}}{\mu}-\frac{3\mu_\theta^2}{\mu^2}+\frac{\mu_\theta}{\mu}\frac{\cos\theta}{\sin\theta}\right]=\frac{k^2\beta}{\mu^2},$$

$$\frac{\mu_{l\theta}}{\mu}-\frac{\mu_\theta}{\mu}\frac{\cosh l}{\sinh l}=0;$$

the rest disappear identically.

For the equations to be mutually consistent, β must have the value $3/k^2$, and the solution is then $\mu=\cosh a\cosh l+\sinh a\sinh l\cos\theta$, that is, $\mu=\cosh\rho_1/k$ where ρ_1 is the distance from a point to be determined by data outside the differential equations.

The geometrical meaning of the equations

$$B_{\mu\nu\sigma\rho}=\alpha(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho})$$

is in all dimensions the same: they are the conditions for congruence.

Applied to three dimensions they lead to the three kinds of space, Euclidean, hyperbolic, and spherical, and no others. Applied to four dimensions they lead similarly to three kinds of space, each associated with a definite system of time measurement, viz. Euclidean space and uniform or Newtonian time, de Sitter's empty world above described, and another form in which the space geodesics and light tracks of de Sitter would be interchanged. In this last, rods would contract, clocks run fast, and light travel quicker as we departed from the centre.

Is there any relative reasonableness in de Sitter's form? I think there is. The basic assumption is that there is some influence, as it were, binding the universe together. This is betrayed in the first place by its effect on test matter: this is most condensed at the centre and becomes more and more diffuse as we depart from that centre. If the universe were filled with matter, and light had to travel through it, it would then travel faster as it departed from the centre, and the connection between the coefficient of expansion and index of refraction would be somewhat as in Poincaré's Model, where the two sets of geodesics are made to agree. But that is not the assumption: the world is empty and light is travelling in vacuo: the same influence which, so to say, loses its grip on matter as we depart from the centre, loses its grip on light also: energy falls off in both cases, and light travels more slowly until, at the boundary, energy becomes so reduced that light stops.

The B equations, as we have seen, lead to definite values of the coefficients in $ds^2=g_{\mu\nu}\,dx_\mu\,dx_\nu$. This leaves no room for variations produced by the presence of matter: for the general theory of Relativity these equations are too stringent. In two and three dimensions the equations $G_{\mu\nu}=\beta g_{\mu\nu}$ are fully equivalent to the B equations, and therefore have the same geometrical meaning. In four dimensions they are not the equivalent of the B equations; presumably they have some geometrical meaning, but I cannot find out what it is. However that be, they still exist, and they are the simplest equations that have the necessary characteristic of permanence—of being independent, that is, of any particular system of coordinates. It is therefore inevitable to take them at least tentatively, as has been done, as the basis for the theory of gravitation.

W. C. Fletcher.

THE CUBIC EQUATION AS A RELATION BETWEEN COMPLEX VARIABLES.

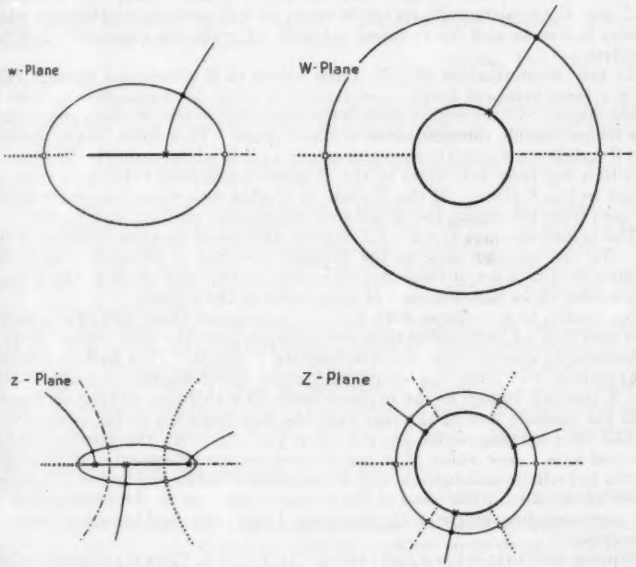
By Professor E. H. Neville.

Interest in the trick that we all know was not discovered by Cardan is greatly enhanced if we think less of a cubic equation and more of a three-one relation between complex variables. It is convenient to take the relation in the form

$$8z^3 - 6z = 2w,$$

and Cardan's solution consists in analysing this into the three simpler relations:

$$2z = Z + 1/Z, \quad Z^3 = W, \quad W + 1/W = 2w.$$



To a given value of w correspond two values of W, say W' and W'', but each of these is the reciprocal of the other, the three values of Z derived from W' are the reciprocals of the three values of Z derived from W'', and the three values of z derived by way of Z from W' are therefore identical with the three values of z derived by way of Z from W''. In other words, although the w-plane is represented twice on the W-plane, it is immaterial to our present purpose whether we recognise the double representation or attend in the W-plane only to some region with which the w-plane has a one-one correspondence.

The geometry of the relation

$$W + 1/W = 2w$$

is familiar. The unit circle in the W-plane corresponds to the part of the real w-axis between $+1$ and -1, and if the w-plane is cut along the line between these points, the remainder of the plane is in point-to-point correspondence with the part of the W-plane outside the circle. To a W-circle

with centre at the origin and radius r corresponds a w-ellipse with foci at ± 1 and semi-axes $\frac{1}{2}(r \pm 1/r)$. In the region outside the unit circle in the W-plane, the orthogonal trajectories of the circles described about the origin are half-lines which produced inwards would pass through the origin; in the w-plane cut between $+1$ and -1, the orthogonal trajectories of the ellipses whose foci are ± 1 are half-branches of hyperbolas with the same foci. To a radial half-line in the W-plane corresponds in the w-plane the hyperbolic half-branch which goes to infinity in the same direction as the half-line; if the half-line in the W-plane makes an angle α with the positive half of the real axis, then since the foot of the perpendicular from a focus on an asymptote is on the auxiliary circle, the transverse semi-axis of the corresponding hyperbola is $\cos \alpha$, and the vertex of the half-branch is at the point $\cos \alpha$ in the w-plane. As the region outside the unit circle in the W-plane is swept by a rotating half-line, the w-plane with its cut is swept by a hyperbolic half-branch whose vertex moves to and fro between $+1$ and -1 while the asymptotic half-line revolves.

In the transformation $Z^3 = W$, three values of Z correspond to each value of W; these values of Z are the corners of an equilateral triangle whose centre is the origin. To the region outside the unit circle in the W-plane corresponds the region outside the unit circle in the Z-plane. To a circle whose centre is the W-origin correspond three equal arcs of a circle whose centre is the Z-origin, and to a ray from the origin in the W-plane correspond three rays from the origin in the Z-plane. If the Z-plane is divided into three congruent sectors by rays from the origin, the W-plane is represented once on each sector.

The transformation $2z = Z + 1/Z$ repeats the transformation between w and W. To the circular arcs in the Z-plane correspond elliptical arcs in the z-plane, and to a set of three half-lines outside the unit circle in the Z-plane correspond three half-branches of hyperbolas in the z-plane.

So, finally, to a w-ellipse with foci ± 1 correspond three arcs of a z-ellipse, also with foci ± 1 but smaller than the w-ellipse, since the intermediate Z-circle is necessarily smaller than the intermediate W-circle. To a half-branch of a w-hyperbola with foci ± 1 correspond three half-branches of z-hyperbolas, and if the half-branch in the w-plane tends to a half-line making an angle α with the positive half of the real axis, the half-branches in the z-plane tend to half-lines making angles $\frac{1}{3}\alpha$, $\frac{1}{3}(\alpha + 2\pi)$, $\frac{1}{3}(\alpha - 2\pi)$ with the positive half of the real axis. Any value of w can be seen as the intersection of an ellipse with a hyperbolic half-branch, and if the relation between z and w is regarded as an equation in z, the roots of the equation are seen as the intersections of the corresponding ellipse with the three hyperbolic half-branches that are appropriate.

Suppose now that w has a real value c. If $|c| > 1$, there is an undegenerate w-ellipse through c, and if c is positive the hyperbolic half-branch through c is the part of the real axis beyond $+1$. There are then definite circles in the W-plane and the Z-plane, outside the unit circles, but the half-line in the W-plane is part of the real axis, and of the three half-lines in the Z-plane, one lies along the positive half of the real axis while the others are inclined at angles of $\pm \frac{2}{3}\pi$ with that one. Hence when we reach the z-plane, we find an undegenerate ellipse, and the z-points corresponding to c are the points in which that ellipse is cut by the positive half of the real axis and by that branch of the hyperbola $4x^2 - \frac{4}{3}y^2 = 1$ which has its vertex on the negative half of the real axis; the hyperbola does not depend on the value of c.

If $|c| < 1$, the w-ellipse degenerates to the line between the foci, the W-circle and the Z-circle are the unit circles, and the z-ellipse also degenerates to the line between the foci. It is in this case the hyperbolic half-branches that vary with c, and to solve the equation we are directed to draw a half-branch with vertex at c and foci ± 1, to find the angle α made by the asymptotic half-line of this curve with the initial line, to draw in the z-plane half-lines

making angles $\frac{1}{3}a$, $\frac{1}{3}(a+2\pi)$, $\frac{1}{3}(a-2\pi)$ with the initial line, and to find the vertices of the hyperbolic half-branches to which these half-lines are asymptotic. Wanting only the vertices of the several half-branches, we need only relate the positions of these vertices to the angles made by the asymptotic half-lines, without making mention of the curves. But we have already noticed that $c=\cos a$, and therefore not only do we understand the intrinsic difference, for a real value of w, between the case when $|w|>1$ and the case when $|w|<1$, but we are led directly from Cardan's solution to the elementary trigonometrical solution that is usual in the "irreducible" case.

E. H. NEVILLE.

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423. Letter exhibited by Mr. Latimer Clark at the conversazione given to Helmholtz, at University College, in 1881:

London, Dec. 15, 1716.

DEAR DOCTOR:

He that in ye mine of knowledge deepest diggeth, hath, like every other miner ye least breathing time, and must sometimes at least come to terr: alt for air.

In one of these respiratory intervals I now sit doune to write to you, my friend.

You ask me how, with so much study, I manage to retene my health. Ah, my dear doctor, you have a better opinion of your lazy friend than he hath of himself. Morpheous is my best companion; without 8 or 9 hour of him yr correspondent is not worth one scavenger's peruke. My practizes did at ye first hurt my stomach, but now I eat heartily enow as y'will see when I come down beside you.

I have been much amused by ye singular p[henomenon] resulting from bringing of a needle into contact with a piece of amber or resin fricated on silke clothe. Ye flame putteth me in mind of sheet lightning on a small—how very small—scale. But I shall in my epistles abjure Philosophy whereof when I come down to Sakly I'll give you enow. I began to scrawl at 5 mins frm 9 of ye clk, and have in writing consmd 10 mins. My Ld. Somerset is announced.

Farewell, Gd. bless you and help yr sincere friend

To Dr. Law, Suffolk. (Signed) ISAAC NEWTON.

424. Feb. 1716, Lady Cowper, Lady of Bedchamber to P. of Wales, in diary: Sir Isaac Newton and Dr. Clarke came this afternoon to explain Sir Isaac's system of philosophy to the Princess.

425. I have heard . . . that Sir Isaac used to say, that Infidelity would probably prevail until it had quite banished superstition, but then would be swallowed up by the great Light and Evidence of True Religion.—Hartley to Stukeley on Newton and Prophecies: Nichol, ii. p. 804, 1734, and Jan. 1, 1735-6.

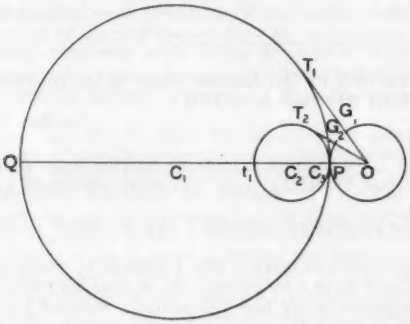
426. Edward Fitzgerald wrote to Pollock recommending the substitution, at the foot of Newton's statue, instead of Lucretius' lines (which might apply to Shakespeare, etc.), of the first words of the *Novum Organum*: *(Homo) Naturae, Minister & Interpres*, which connects him with Barrow, who sits along with him.

427. A memorandum in the handwriting of Sir William Jones on the fly-leaf of Newton's *Principia* (1713): Burrow [Reuben] told me that he had seen in Newton's handwriting opposite (in list of mathematical books) to my father's Synopsis, "multum in parvo" or some such phrase: TAFAZZUL HUSAIN says Burrow told him the phrase was "an ocean in a pitcher."

MATHEMATICAL NOTES.

866. [V. 1. a. λ.] *A Graphical Introduction to Logarithms.*

The properties of the table of successive square roots given in Dr. Miller's article on the introduction of *e* (*Gazette*, xiii. p. 104) may be shown by the following simple construction, while the limit process which gives rise to the logarithms appears with greater generality than any arithmetical table can give.



Construction. Draw a circle with centre O and unit radius.

Continue OP to Q, so that $OQ=n$ and on PQ as diameter draw circle Σ_1 centre C_1.

Draw OT_1 the tangent to Σ_1 and make $Ot_1=OT_1$.

On Pt_1 as diameter draw circle Σ_2 centre C_2 and draw OT_2 tangent to Σ_2.

Continue the process to C_m.

By elementary geometry $\quad OT_1^2=OP\,.\,OQ$

$$=1\,.\,n,$$

and $\quad OT_2^2=1\,.\,OT_1=n^{\frac{1}{2}}.$

Clearly the sequence $C_2, C_3 \ldots C_m$ approaches P, the limit being a point circle at P, while at the same time $OT_m \to OP$, *i.e.*

$$\lim_{m\to\infty} n^{1/2^m}=1.$$

Moreover, if $OT_1, \ldots, OT_m$ cut the unit circle in $G_1, \ldots, G_m$. $T_1G_1 \ldots T_mG_m$ are logarithms of n.

If $\quad T_1G_1=x_1 \quad$ and $\quad T_2G_2=x_2,$

then $\quad x_1+1=n^{\frac{1}{2}}=(x_2+1)^2;$

$$\therefore\ x_1=2x_2+x_2^2.$$

Hence if $n>1$ we have generally $\dfrac{x_{m-1}}{x_m}=2+x_m$, while

$$\lim_{m\to\infty} \frac{x_{m-1}}{x_m}=2.$$

The diagram is drawn for the case $n=9$. If $n<1$ the diameter of circle Σ_1 is $(n-1)$, a negative quantity, which must be measured from P towards O.

The circles approach P in the same way, while at the same time

$$x_1 = 2x_2 - x_2^2.$$

Hence if $n < 1$, $\lim_{m\to\infty} \frac{x_{m-1}}{x_m} = 2$ (approaching from values below 2).

Exercises in geometrical drawing may be based on this construction as follows:

(*a*) Obtain for a pair of numbers, say 2 and 3, the successive logarithms and add together those of the same order.

(*b*) Using these values (the same order logarithms of 6), invert the construction given and obtain a sequence of circles which in limit must be the circle for $n = 6$. H. PEAT.

867. [K¹. 7. e.] *Distance-Relations in an Involution Range.*

In the text-books (*e.g.* J. W. Russell, *Pure Geometry*), the following four relations are given connecting the distances between points forming the involution range (AA', BB', CC'), viz.:

$$AB' \cdot BC' \cdot CA' = -A'B \cdot B'C \cdot C'A,$$
$$AB' \cdot BC \cdot C'A' = -A'B \cdot B'C' \cdot CA,$$
$$AB \cdot B'C' \cdot CA' = -A'B' \cdot BC \cdot C'A,$$
$$AB \cdot B'C \cdot C'A' = -A'B' \cdot BC' \cdot CA.$$

These are, of course, easily verifiable separately from the definition of an involution range, but I have not seen any coherent geometrical proof of all four, nor indeed any *geometrical* reason why there should be just four such relations. The following discussion, I think, throws a little more light on the matter.

Four intersecting straight lines forming a complete quadrilateral determine four triangles, and each line may be regarded as a transversal to the sides of the triangle formed by the other three. Taking a triangle $\alpha\beta\gamma$ with a transversal meeting $\beta\gamma$ at α', $\gamma\alpha$ at β', $\alpha\beta$ at γ', we have the "Menelaus ratio"

$$\frac{\alpha\beta'}{\beta'\gamma} \cdot \frac{\beta\gamma'}{\gamma'\alpha} \cdot \frac{\gamma\alpha'}{\alpha'\beta} (= -1),$$

and three similar relations for the other three triangles. But it is easily proved that such a "Menelaus ratio" is projective, for it is replaceable by the expression

$$\frac{\sin \alpha V\beta'}{\sin \beta' V\gamma} \cdot \frac{\sin \beta V\gamma'}{\sin \gamma' V\alpha} \cdot \frac{\sin \gamma V\alpha'}{\sin \alpha' V\beta},$$

where V is any point in the plane.

Hence, by projecting on to any straight line, the four relations emerge at once as a consequence of the well-known theorem that the connectors of the vertices of a complete quadrilateral to any point in its plane form a pencil in involution.

15th May, 1926. E. H. SMART.

868. [K¹. 2. a. c.] *Lemoine's (?) Theorem.*

Lemma. If PQ is any diameter of circle ABC and AU is perpendicular to PQ, then the pedal lines of P, Q for $\triangle ABC$ meet at ω on the nine-point circle, ω being the image of U in $B'C'$.

(Proved before.)

Let S, S' be isogonal conjugates for $\triangle ABC$, *i.e.* foci of inscribed conic. Let the perpendiculars SL, $S'L'$ meet the line through A parallel to BC in l, l' respectively.

Draw AU, AU' perpendicular to the diameters OS, OS' of circle ABC.

Then, from the cyclic quadrilateral $Al'U'S'$,

$$\angle U'l' = \angle U'S'A = 90° - \angle U'AS'.$$

Also $\angle U'US = \angle U'AO$ from quadrilateral $AUOU'$

and $\angle lUS = 180° - \angle lAS$ ..... $AlUS$;

$$\therefore \angle lUU' = 180° - \angle lAS - \angle U'AO$$
$$= 180° - \angle lAS - \angle S'AO + \angle U'AS'$$
$$= 180° - \angle lAS - \angle SAH + \angle U'AS'$$
$$= 90° + \angle U'AS';$$

$\therefore$ the four points U, U', l', l lie on a circle.

Reflect this circle in the line $B'C'$ joining the middle points of AC, AB; then L, L', ω, ω' are concyclic, where ω, ω' are the poles of diameters OS, OS'.

Similarly, working from B, M, M', ω, ω' are concyclic,

and so N, N', ω, ω' are concyclic.

But the six points L, M, N, L', M', N' lie on a circle (the auxiliary circle).

Hence, either the four lines LL', MM', NN', $\omega\omega'$ are concurrent,

or the eight points ω, ω', L, L', M, M', N, N' are concyclic;

but the first alternative is impossible, since LL', MM', NN' form the $\triangle ABC$, so the auxiliary circle (the common pedal circle of S and S') passes through ω and ω'.

Cor. I. When S, S' coincide, so do ω, ω'; hence the N.P.C. touches the incircle and the three escribed circles.

Cor. II. As the point S moves along the diameter OS, the corresponding pedal circles all intersect the N.P.C. in the fixed point ω.

1922. E. P. LEWIS.

869. [A. 1. b.] *The relation of the A.M. to the G.M.*

The following is a simple demonstration, of which I do not know the author, of the familiar theorem of which a transcendental proof was given recently in the *Gazette* (p. 244 of the current volume).

Let $a_1, a_2 \dots a_n$ be all positive, and let

$$f \equiv (x - a_1y)(x - a_2y) \dots (x - a_ny)$$
$$\equiv \sum_{r=0}^{n} (-)^r \cdot {}^nC_r \cdot p_r x^{n-r} y^r.$$

Then $p_1, p_2 \dots p_n$ are necessarily positive, and

$$np_1 = a_1 + a_2 + \dots + a_n,$$
$$p_n = a_1 a_2 \dots a_n.$$

Since f has none but real roots in x/y, the derivatives of f will have none but real roots in x/y.

The derivatives of order $(n-2)$ are quadratics, which, after discarding a numerical factor, may be written

$$x^2 - 2p_1xy + p_2y^2, \quad p_1x^2 - 2p_2xy + p_3y^2, \dots, \quad p_{n-2}x^2 - 2p_{n-1}xy + p_ny^2.$$

Since these have real roots,

$$p_1^2 > p_2, \quad p_2^2 > p_1p_3, \dots, \quad p_{n-1}^2 > p_{n-2}p_n;$$

$$\therefore p_1 > \frac{p_2}{p_1} > \frac{p_3}{p_2} > \dots > \frac{p_n}{p_{n-1}};$$

$$\therefore p_1^n > p_1 \cdot \frac{p_2}{p_1} \cdot \frac{p_3}{p_2} \dots \frac{p_n}{p_{n-1}};$$

$\therefore p_1^n > p_n$, which is the A.M. > G.M. result.

Obviously the method leads to many less known inequalities.

S. L. GREEN.

REVIEWS.

Projective Geometry. By C. V. DURELL. Pp. x+222. 8vo. 7s. 6d. 1926. (Macmillan.)

This excellent work is an abridged form of the author's *Course of Plane Geometry for Advanced Students*, Part II. 1910, and the principles of general projection with real and imaginary elements are clearly set before the reader at an early stage.

The book covers most of the subjects treated in the earlier work, and as might be expected the proofs are in many cases very condensed, so that it would hardly be suitable for beginners, *e.g.* on page 68, line 9, and page 69, line 9, there should be some reference to the special meaning and consequences of common corresponding rays and points; but for those who have mastered the former treatise the present one would admirably serve the purpose of revision, and advanced readers would find pleasure in turning over the pages and renewing their acquaintance with the many properties of conics which have been discovered of late years.

As the author deals so freely with the extension to conics of properties of the circle, it is rather strange that he has omitted to extend the theory of inversion to the conic. The three fundamental propositions are:

(1) The inverse of a straight line with respect to a conic is a conic passing through six points, viz. the pole of inversion, the pole of the given line, and the points where their polars intersect the conic of inversion.

(2) The inverse of a conic passing through the pole of inversion T and the points I, I', where its polar intersects the conic of inversion is a straight line, viz. the other common chord corresponding to II'.

(3) If A is the conic and T the pole of inversion, the inverse of a conic B, which passes through I and I', but not through T, is a conic C of the pencil determined by A and B.

Simple direct proofs of these and other properties are given in the *Mathematical Gazette* for July, 1919.

At intervals through the book are given interesting historical notes which enable the student to follow the various stages in the development of the subject, but these notes, to be of value, should be accurate. There is rather a tendency to attach to any important property the name not of the discoverer, but of the writer who makes use of it, *e.g.* the *locus ad tres et quatuor lineas*, which, as is common knowledge, was known to, and solved partially by Euclid, 300 B.C. and completely by Apollonius, is referred to as Pappus' theorem, 400 A.D. Again the important property which Apollonius tells us he discovered and which enabled him to obtain the complete solution of the locus is called Newton's theorem. Certainly Newton makes great use of it in Sec. V., Bk. I. of his *Principia*, but the first time he employs it (Lemma, xvii.) he carefully states that it is Props. 17, 19, 21, 23, Bk. III. of Apollonius' *Conics*.

Again the statement on p. 90 that Pascal proved the Mystic Hexagram for the circle is not quite correct. A reference to the *Mathematical Gazette* for March, 1924, would show that he put it forth without proof. The particular case in which the conic degenerates into two straight lines is often spoken of as Pappus' theorem, because he gave a solution of it, but it would be just as logical to call the *locus ad tres et quatuor lineas* by the name of Descartes or Newton, because they were supposed to be the first to obtain a solution of it, the one by algebraical and the other by pure geometry. Of course we now know that the solution which was supposed to be lost is to be found in the propositions at the end of Ap. Bk. III.

The only errata noted are:

P. 27, line 23, *for* Q *read* P.
P. 77, fig. 37, *for* D, K *write* D', K'.
P. 92, line 14, *for* AD *read* A, D.
P. 157, line 21, *for* tangent *read* tangents.

JOHN J. MILNE.

Preliminary Geometry. By H. McKay. Pp. viii+112. 2s. 1926. (Oxford University Press.)

This is a graduated set of exercises intended to lead up to a course of deductive geometry, and is based largely upon drawing and measurement. Though the meaning is always intelligible to the teacher, some parts will scarcely be so to the children for whom they are intended. A vagueness of expression, perhaps intentional, pervades the book, and portions, at least, seem to be of but little value. It is well printed, and has abundance of diagrams, but several mistakes are noticeable.

Theoretical Geometry. By A. W. Siddons and R. T. Hughes. Pp. xvi+173. 3s. 1926. (Cambridge University Press.)

This admirable textbook is based on the various geometry books by Godfrey and Siddons, the numerical and practical parts being omitted, so that the whole logical structure is not obscured. Briefly the course is in the following order—angles and parallels, congruent triangles, inequalities, parallelograms, intercepts by parallel lines, rectilinear areas, Pythagoras' theorem and its extensions, the two fundamental loci, properties of circles, proportional division of straight lines, similar triangles, and a short introduction to more advanced plane geometry, followed by a few riders on solid geometry. There is no attempt to develop the theory of solid geometry.

The book is extremely well written, the language clear and precise, and there is abundance of riders, carefully graduated and grouped. The various topics are arranged in chapters; it is unfortunate that the heading of Chapter XI. has dropped out of the list of contents. In the chapter on loci the two aspects, inclusive and exclusive, stand out very clearly. The difficulties inherent in any system of theoretical geometry are honestly faced and expounded as they arise, and there are given many valuable hints to the teacher.

The proofs of the converse theorems giving angle conditions which make four points concyclic are admitted by the authors to be incomplete, and suggestions are made for completing them. Is the following direct method of proof too difficult? Let A and B be two fixed points in a plane, and P a movable point in the same plane, restricted to one side of the line AB, such that the angle APB is of constant magnitude a. For all positions of P a circle can be circumscribed to APB, and the centre of this circle is so related to the points A and B that it has a unique position for all positions of P. Again, let Q be a movable point in the same plane, restricted to the other side of the line AB, such that the angle AQB is of constant magnitude b. Then for all positions of Q a circle can be circumscribed to AQB, and when a and b are supplementary, the centre of this circle has the same position as before.

A Second Geometry. By J. Davidson and A. J. Pressland. Pp. 128. 2s. 6d. 1926. (Oxford University Press.)

This book is a sequel to *A Primer of Geometry*, by W. Parkinson and A. J. Pressland. Half of it is intended to serve as an introduction to the more advanced topics of plane geometry, such as similarity, properties of triangles, coaxal circles, inversion, involution and harmonic ranges, poles and polars. The other half deals with solid geometry. The proofs are generally short and sometimes elegant, but the language is often lacking in precision, and, at times, not easy for a pupil to follow. In naming two similar triangles the authors are not careful to give the letters in corresponding order.

Several errors require to be corrected—*e.g.* "FCD" for "FBD" on page 33, two "k's" on page 45, with different meanings, "pole of P" for "pole of PK" on page 60; in the figure of the regular icosahedron on page 95 one of the thirty edges is omitted. But apart from such errors the book could be considerably improved by more careful editing. Two such consecutive sentences as "Produce OH to P so that $OH \cdot OP = OB^2$. Let the tangents at B and C intersect at P" (page 60), are inexcusably bad geometry, even though the two constructions for P are consistent.

In their short preface the authors suggest that in plane geometry formal treatment is no longer necessary, but that in solid geometry formal treatment

is likely to inspire confidence. So, in beginning solid geometry, they say, "The time has now come when formal definitions must be considered." On the next page they define a horizontal plane as "one parallel to the horizon." After proving the fundamental theorem 95, that if AB is perpendicular both to BC and to BD, then AB is perpendicular to any line through B in the plane BCD, the following note is added—"A straight line normal to a plane is perpendicular to every straight line in the plane, not merely to those that pass through the foot of the normal." This corollary is quoted later in proving theorem 100, though it is not until after theorem 108 has been proved that "the angle between two skew lines" is, or can be, considered.

The "inclination of two planes," leading immediately to propositions concerning two perpendicular planes, is discussed on page 76, and is so obviously out of place that reference has to be made to a theorem proved on page 85.

Theorem 113 proves that the joins of the middle points of the opposite sides of the tetrahedron $A(BCD)$ are concurrent and bisect each other in O. On the next page theorem 114 proves that the joins of the vertices of the tetrahedron $A(BCD)$ to the centroids of the opposite faces are concurrent in K. How many pupils, reading this book, would imagine that O and K are one and the same point?

W. J. DOBBS.

Modern Puzzles. By H. E. DUDENEY. Pp. 185 and Index. 3s. 6d. 1926. (C. Arthur Pearson, Ltd.)

A recent English book on Denmark, commenting on the remarkable success of Danish agriculture, attributed its striking prosperity to a cause which is of very deep significance to the mathematician as to every other technical expert. England, according to this author, produces practical men every bit as good as the practical Dane, and theorists at least as good as the Danish theorist—possibly even better. Why then are we behind our neighbours? Simply, he suggests, because in Denmark the man on the land and the man in the laboratory trust each other and work together, while in England there is misunderstanding, contempt and even hostility between the theorist and the man of action. "Something is rotten in the state of—" not Denmark this time but nearer home.

Can we deny this charge? Or must we rather admit that Agriculture is only a narrow sector of the wide front on which this grievous national fault is hampering our advance? In Music or in Painting is there any anomaly which impresses itself so deeply on a thoughtful observer as the profound gulf which separates the expert's idea of beauty from the ordinary "bloke's"? Do our leaders of thought lead? Who understands philosophers? Do the treasurers of our national wisdom carefully sift their treasure in an unceasing attempt to render the maximum of it available to their inferiors in their own particular line? Why does the man in the bus class Mathematics as bosh? And whose fault is it?

In Mathematics, as in every branch of study, one of the country's most vital needs is the bridging of the gulf between the mathematician and the bloke, the man of sound common sense and active mind, but intolerant of the bushel of technical terms beneath which professional thinkers hide their light—a jargon which he suspects the expert of having invented from pure professional snobbery to exalt his own prestige, as the majesty of the law is emphasized by obscure legal phrases. I suggest that if the *Mathematical Gazette* has the enterprise to organize a post-card vote among the entire national blokage (perhaps "one who doesn't read the *M.G.*" would make a serviceable definition) to enquire what mathematician has shed most light and joy, the answer will not be "Albert Einstein" but "Henry Dudeney." And would it be horribly profane to suggest that this answer would be right? And if it would, then why not be horribly profane?

This is a long prelude to a review: perhaps it is justified if it serves to outline a view of life, a standpoint from which Mr. Dudeney's work is of real importance. If the gulf that separates the outlook of the *Mathematical Gazette* from that of the *Strand Magazine* is a national danger, then one of the hopes of national salvation is the man who can set to work to bridge it.

Mr. Dudeney's books of puzzles, mostly reprinted from the *Strand Magazine* and other light papers, show him as an author singularly endowed with the faculty of reconciling Mathematics with blokery. Such a man must have a long reach both up and down, up into Mathematics more advanced than the *Strand Magazine* would ever guess, and down into tosh more light-hearted than the *Mathematical Gazette* would ever countenance. Coupled with this must be, and is, an imagination of astonishing fertility, and a clarity of explanation in common English words which should be an example to many of our writers of text-books.

The well is so full of water that a few bucketfuls may be drawn from it without danger of depletion. Two "missing figure" problems (70 and 71), one of which I believe has appeared in the *Mathematical Gazette*, are of singular beauty. They admit of solution without ambiguity and without successive trial of alternative possibilities: yet in one of them only one figure is given and forty are to be found, while in the other forty-six are asked for and no figures at all are given—only one decimal point (and that a small one).

Most mathematicians will expect to be able to do this in their heads, but not so many will succeed. (38). "If an army forty miles long advances forty miles while a despatch-rider gallops from the rear to the front, delivers a despatch to the commanding general, and returns to the rear, how far has he to travel?" And can you express 64 by means of two fours and arithmetical signs? (58).

The Four-colour Theorem, pronounced insoluble by some mathematicians, is discussed in the solutions (140); and a demonstration is given about which, as Mr. Dudeney tells us, the opinions of mathematicians differ whether it is a valid proof or not.

Many of the geometrical questions are distinctly difficult. Can you cut a regular hexagon into five pieces that will fit together and form a square? (108). I can; but that is because I have looked up the answer in the crib at the end of the book. Number 124 raises a curious point. If Euclid doesn't allow the edge of a saucer as an instrument for drawing a circle with, why does he allow a ruler for drawing a straight line? And if no ruler or stretched string is admissible, can a straight line still be drawn? No doubt we will be told that there is no such thing as a straight line by those who can prove that there is no such thing as anything; but Mr. Dudeney's solution is more satisfying than this.

180 gives a pleasant special case of a problem for which no general solution has yet been found. Here is a challenge to someone in quest of adventure.

Some of the mathematical games are most entertaining and none-too-easy of analysis though the rules are absurdly simple. Finally a special mention is due to an admirable question (201) in which both sides' scores in six matches are deduced from data at first sight much too scanty to determine twelve unknown quantities.

Mistakes are rare: such as do appear are mostly minor errors of approximation (38, 209). A cube whose "superficial area equals the square of the cubical contents" will make some readers shiver (96); and in 32 the author treats the velocities of a cyclist with and against the wind by the additions and subtractions which would be appropriate for an aeroplane. Though there is no theoretical justification for this, it would be interesting to know whether experiment has proved it a serviceable approximation.

For the "highbrow" who wishes to despise the interests of less advanced students than himself there is much in these pages that will gratify his taste for contempt. Those, on the other hand, who are deeply concerned to arouse in the not-particularly-mathematical a taste for a subject which often repels them if presented in too stodgy a form, will hail Mr. Dudeney's works, and *Modern Puzzles* not the least of them, as pearls of great price, or as mines of interesting and cram-proof questions for pupils. And to the student of human affairs who deplores the divorce of mind between the Pharisee and Publican, this book should appeal as a truly valuable contribution towards the bridging of the gap and the establishment of a common field of interest and thought.

W. HOPE-JONES.

REPORT OF THE GIRLS' SCHOOLS COMMITTEE FOR THE YEAR 1926.

DURING the year the Committee have considered the following subjects:

(1) The Teaching of Ratio and Proportion.

(2) The Teaching of Arithmetic in Elementary Schools with a view to better understanding and cooperation between Elementary and Secondary Teachers.

(3) Newnham and Girton Scholarship Papers in Mathematics.

With regard to (2), Miss Lawrence of Cannonbury Road School came to a meeting of the Committee and told about the organisation and work done in Arithmetic in her school. The general discussion was very valuable. Various members of the committee are trying to get to know more of the work in Elementary Schools in their own districts, with a view to better cooperation.

The consideration of (3) was undertaken at the request of some teachers of Mathematics in Girls' Schools, who felt the Scholarship papers were growing too difficult and meant undue strain for girls at school.

The matter was discussed with Miss Swain, Mathematical Tutor at Newnham College.

M. A. HOOKE, *Hon. Sec.*

REPORT OF THE QUEENSLAND BRANCH FOR THE YEAR 1925.

President—Professor H. J. Priestley, M.A.
Vice-Presidents—Mr. S. Stephenson, M.A.; Mr. R. A. Kerr, M.A., B.Sc.
Hon. Secretary and Treasurer—Mr. J. P. M'Carthy, M.A.
Committee—Miss D. E. Lockington, B.A.; Miss C. M. Cribb, M.A.; Mr. I. Waddle, M.Sc.; Mr. H. J. Priest, B.A., B.Sc.; Mr. E. W. Jones, B.A.

FOURTH ANNUAL REPORT OF THE BRANCH: PRESENTED TO THE ANNUAL GENERAL MEETING HELD AT THE UNIVERSITY ON MARCH 26TH, 1926.

Ladies and Gentlemen,—I have the honour to present to you the Fourth Annual Report of the Queensland Mathematical Society. The Annual Meeting was held at the University on March 27th, 1925. The Annual Report and Balance Sheet were adopted by the meeting. After the election of officers for the ensuing year Professor Priestley delivered his presidential address, taking as his subject, "Modern Geometry." During the year three general meetings have been held, at the first of which, held on May 15th, Mr. E. W. Jones, B.A., spoke on "The Correlation of Trigonometry and Coordinate Geometry." The second meeting held on July 31st took the form of a discussion on "The Theory of Limits and Convergency," and at the third, held on October 30th, Mr. S. G. Brown, M.A., B.Sc., addressed the meeting on "The Teaching of Elementary Algebra."

The number of financial members of the Branch for the year was twenty-three, of whom thirteen are full members of the Mathematical Association: the membership thus remains practically unchanged from last year. There are, however, quite a number of teachers of mathematics and of persons interested in the subject who are not members of the Branch: perhaps this coming year will bring an increase in membership. Associate members have received copies of the *Gazette* from time to time.

Financially the Branch is in a healthy condition: the balance sheet shows a credit balance of £3 1s. 9d., which is the best since the foundation of the Branch.

The good attendances at meetings during the past year promises well for the year to come.—I have the honour to be, Ladies and Gentlemen,

J. P. M'CARTHY, *Hon. Sec. and Treasurer.*

REPORT OF THE SYDNEY BRANCH FOR THE YEAR 1926.

THE membership now stands at 28 full members, and 80 associate members. The work of distribution of the *Gazette* among the associate members has proceeded satisfactorily.

There have been two meetings of the Association during the year. At the first meeting, Mr. Colville gave an interesting address on "Some Experiences in Teaching the Early Work in Algebra." This was much appreciated, but unfortunately no discussion could take place, because of lack of time. At the same meeting Prof. Wellish spoke on "Vector Methods," and showed how these could be used with advantage in certain parts of geometry and mechanics, with the students doing higher work.

At the Annual Meeting, Dr. Brose spoke on "Some Aspects of Relativity." Dr. Brose is an old Rhodes scholar of Adelaide, who has become well known as the translator of some of Einstein's and Weyl's works, as well as for his own writings; and his presence in Sydney for this year gave the Association an opportunity of hearing him. His address was much appreciated.

The Office-Bearers for 1927 were elected as follows:

President, Prof. H. S. Carslaw; *Honorary Treasurer*, Mr. A. C. Nairn; *Joint Honorary Secretaries*, Miss J. Brown and Mr. H. J. Meldrum.

CORRESPONDENCE.

An Introduction to Mechanics. By J. P. Clatworthy.

DEAR SIR,

I very much regret that one of the statements in my review of the above book (see *Gazette*, No. 185, Dec. 1926, p. 254) is incorrect; namely that "In Part I., p. 102, Ex. 1, the positions of the forces on the rod are not stated, and have to be inferred from the figure and the proof." As a matter of fact there is a reference in the example to Fig. 62 on p. 100, which belongs to another example on the same rod where the positions of the forces are stated.

J. W. HARMER.

THE LIBRARY.

160 CASTLE HILL, READING.

ADDITIONS.

The Librarian reports gifts as follows:

From Miss **M. H. Greaves** and from Miss **I. M. Lewis**, collections of back numbers of the *Gazette*.

From the Misses **Johnston**, offprints of two papers on quaternions by the late Professor W. J. Johnston.

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